GREEDY POLYOMINOES AND FIRST-PASSAGE TIMES ON RANDOM VORONOI TILINGS

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Abstract. Let $\mathcal{N}$ be distributed as a Poisson random set on $\mathbb{R}^d$ with intensity comparable to the Lebesgue measure. Consider the Voronoi tiling of $\mathbb{R}^d$, $\{C_v\}_{v\in\mathcal{N}}$, where C_v is composed by points $\mathbf{x} \in \mathbb{R}^d$ that are closer to $v \in \mathcal{N}$ than to any other $v' \in \mathcal{N}$. A polyomino $\mathcal{P}$ of size n is a connected union (in the $\mathbb{R}^d$ topological sense) of n tiles, and we denote by Π_n the collection of all polyominos $\mathcal{P}$ of size n containing the origin. Assume that the weight of a Voronoi tile C_v is given by $F(C_v)$, where F is a nonnegative functional on Voronoi tiles. In this paper we investigate for some functional F, mainly when $F(C_v)$ is the number of faces of C_v, the tail behavior of the maximal weight among polyominoes in Π_n: $F_n = F_n(\mathcal{N}) := \max_{\mathcal{P}\in\Pi_n} \sum_{v\in\mathcal{P}} F(C_v)$. Next we apply our results to study self-avoiding paths, first-passage percolation models and the stabbing number on the dual graph, named the Delaunay triangulation. As the main application we show that first passage percolation has at most linear variance.

1. Introduction

Greedy animals on $\mathbb{Z}^d$ have been studied notably in Cox et al. (1993). Imagine that positive weights, or awards, are placed on all vertices of $\mathbb{Z}^d$. A greedy animal of size n is a connected subset of n vertices, containing the origin, and which catches the maximum amount of awards. When these awards are random, i.i.d, it is shown in Cox et al. (1993) that the total award collected by a greedy animal grows at most linearly in n if the tail of the award is no too thick. This is shown in a rather strong sense, giving exponential deviations inequalities. This result has already been proved useful for the study of percolation and (dependent) First Passage Percolation, see Fontes and Newman (1993). In section 2, we extend the results of Cox et al. (1993) to some greedy polyominoes on the Poisson-Voronoi tiling, i.e in this paper, the Voronoi tiling based on a Poisson random set on $\mathbb{R}^d$ with intensity comparable to the Lebesgue measure. Our greedy polyominoes use special dependent random awards, in the sense that the award at each Voronoi tile is a function of the number of faces of the Voronoi tile. We believe that our results may be useful to control the geometry of the Poisson-Voronoi tiling. To illustrate this, we shall give in section 3 three applications of our results on greedy polyominoes to some geometric problems on the Poisson-Voronoi tiling. The Poisson-Delaunay graph is the dual graph of

Leandro P. R. Pimentel was partially supported by grants from the Swiss National Science Foundation, Fundação de Amparo a Pesquisa do Estado de São Paulo and The Netherlands Organisation for Scientific Research.

Raphaël Rossignol was supported by the Swiss National Science Foundation grants 200021-1036251/1 and 200020-112316/1.

the Poisson-Voronoi tiling. The first application is the control, on the Poisson-Delaunay graph, of the number of self-avoiding paths of length n starting from the origin (section 3.1). Section 3.2 is devoted to a technical result about Bernoulli First Passage Percolation on the Poisson-Delaunay graph, which is used in our main application. Next, and this is our main application, we prove in section 3.3 that the first passage time in First Passage Percolation has (at most) linear variance on the Poisson-Delaunay graph. Finally, we show in section 3.4 that the stabbing number of the Poisson-Delaunay graph has an exponential tail. The stabbing number is related to the study of simple random walk on this graph.

In the end of this introduction, we give all the needed formal definitions to state precisely our main results on greedy polyominoes.

Voronoi tilings and polyominoes. To any locally finite subset $\mathcal{N}$ of $\mathbb{R}^d$ one can associate a partition of the plane as follows. To each point $v \in \mathcal{N}$ corresponds a polygonal region C_v, the *Voronoi tile* (or cell) at v, consisting of the set of points of $\mathbb{R}^d$ which are closer to v than to any other $v' \in \mathcal{N}$. Closer is understood here in the large sense, and the partition is not a real one, but the set of points which belong to more than one Voronoi tile has Lebesgue measure 0. The collection $\{C_v\}_{v\in\mathcal{N}}$ is called the *Voronoi tiling* (or tessellation) of the plane based on $\mathcal{N}$. From now on, $\mathcal{N}$ is understood to be distributed like a Poisson random set on $\mathbb{R}^d$ with intensity measure μ. We shall always wuppose that μ is comparable to Lebesgue's measure on $\mathbb{R}^d$, λ_d, in the sense that there exists a positive constant c_μ such that for every Lebesgue-measurable subset A of $\mathbb{R}^d$:

$$\frac{1}{c_\mu}\lambda_d(A) \leq \mu(A) \leq c_\mu \lambda_d(A)\ . \tag{1}$$

For each positive integer number $n \geq 1$, a *Voronoi polyomino* $\mathcal{P}$ of size n is a connected union of n Voronoi tiles (Figure 1). Notice that with probability one, when two Voronoi tiles are connected, they share a $(d-1)$-dimensional face. We denote by Π_n the set of all polyominoes $\mathcal{P}$ of size n and such that the origin 0 belongs to $\mathcal{P}$.

Assume that the "weight" of a Voronoi tile C_v is given by $F(C_v) = f(d_{\mathcal{N}}(v))$, where $d_{\mathcal{N}}(v)$ is the number of $(d-1)$-dimensional faces of C_v and f is a positive nondecreasing function on the integers. In this way we define a random weight functional on polyominoes by

$$F(\mathcal{P}) = F(f, \mathcal{N}, \mathcal{P}) := \sum_{v\in\mathcal{P}} f(d_{\mathcal{N}}(v))\ .$$

The maximal weight among polyominoes in Π_n is:

$$F_n = F_n(f, \mathcal{N}) := \max_{\mathcal{P}\in\Pi_n} F(\mathcal{P})\ .$$

A greedy Voronoi polyomino $\mathcal{P}_n$ is a Voronoi polyomino that attains the maximum in the definition of F_n: $F(\mathcal{P}_n) = F_n$.

To state our next theorem, we require the following notation:

Definition 1.1. *Given a nondecreasing function f from $\mathbb{R}^+$ to $[1, \infty)$, define:*

$$g_f(x) := x f(x)\ ,$$

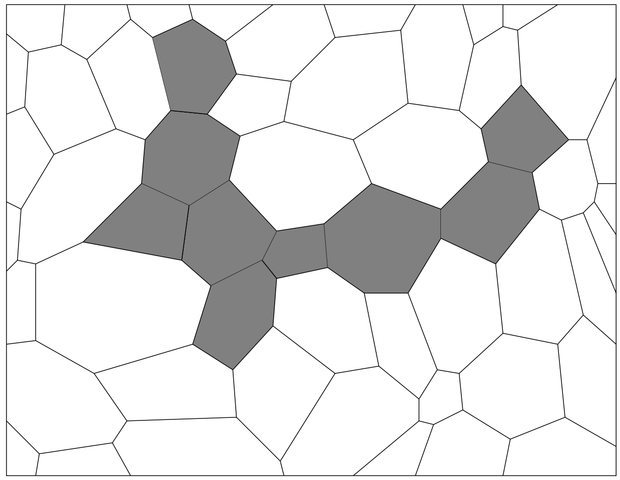

FIGURE 1. A two-dimensional Voronoi polyomino of size $n = 9$.

let g^{-1} denote the pseudo-inverse of any strictly increasing function g:

$$g^{-1}(u) = \sup\{u' \in \mathbb{R} \ s.t. \ g(u') < u\} ,$$

and let:

$$l_f(y) := g_f^{-1}(y) , \ h_f(y) := y l_f(y) , \ q_f(x) := h_f^{-1}(x) .$$

For each $c \in (0,\infty)$, we say that f is c-nice if

$$\liminf_{y\to\infty} \frac{l_f(y)}{\log y} \geq c .$$

Theorem 1.2. *There exists a constant $c_1 \in (0,\infty)$ such that, if f is c_1-nice then there exist constants $z_1, c_2, c_3 \in (0,\infty)$ such that for all $z \geq z_1$ and for all $n \geq 1$:*

$$\mathbb{P}\big(F_n > nz\big) \leq \exp\Big\{ - c_2 l_f\big(q_f(c_3 nz)\big)\Big\} .$$

In particular, there also exists a constant $c_4 \in (0,\infty)$ such that

$$0 \leq f(3) \leq \liminf_{n\to\infty} \mathbb{E}\Big(\frac{F_n}{n}\Big) \leq \limsup_{n\to\infty} \mathbb{E}\Big(\frac{F_n}{n}\Big) < c_4 .$$

The Delaunay graph. An important graph for the study of a Voronoi tiling is its facial dual, the *Delaunay graph* based on $\mathcal{N}$. This graph, denoted by $\mathcal{D}(\mathcal{N})$ is an unoriented graph embedded in $\mathbb{R}^d$ which has vertex set $\mathcal{N}$ and edges $\{u, v\}$ every time C_u and C_v share a $(d-1)$-dimensional face (Figure 2). We remark that, for our Poisson random set, a.s. no $d+1$ points are on the same hyperplane and no $d+2$ points are on the same hypersphere that makes the Delaunay graph a well defined triangulation. This triangulation divides $\mathbb{R}^d$ into bounded simplices called *Delaunay cells*. For each Delaunay cell Δ no point in $\mathcal{N}$ is inside the circum-hypersphere of Δ. Polyominoes on the Voronoi tiling correspond to connected (in the graph topology) subsets of the Delaunay graph . Also, the number of faces of a Voronoi tile C_u, which we denoted by $d_{\mathcal{N}}(v)$ is simply the degree of v in $\mathcal{D}(\mathcal{N})$.

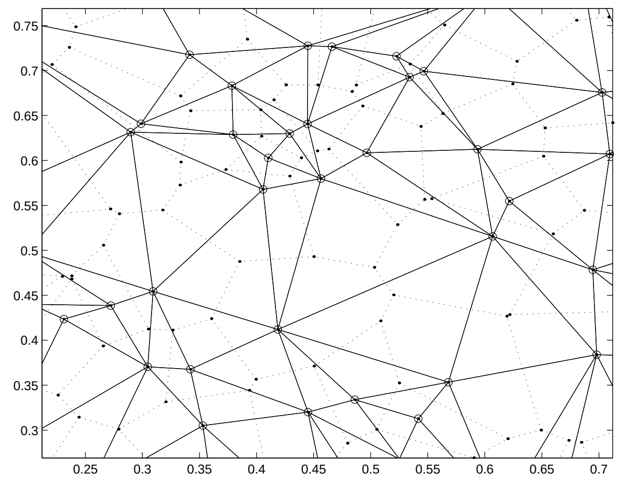

Figure 2. The Voronoi tiling (dashed lines) and the Delaunay triangulation (solid lines) in dimension $d = 2$.

We shall prove an easier variant of Theorem 1.2, which will be useful in the applications. Define Ω to be the set of at most countable subsets of $\mathbb{R}^d$. Then, for any $\omega \in \Omega$ and any subgraph ϕ of $\mathcal{D}(\omega)$, we define $\Gamma(\omega, \phi)$ to be the set of points in $\mathbb{R}^d$ whose addition to ω "perturbs" ϕ:

$$\Gamma(\omega, \phi) = \{x \in \mathbb{R}^d \text{ s.t. } \phi \not\subset \mathcal{D}(\omega \cup \{x\}\} \ .$$

We get the following result for the maximal size of $\Gamma(\omega, \phi)$ when ϕ belongs to $\mathcal{SA}_n$, the set of self-avoiding path on $\mathcal{D}(\mathcal{N})$ starting from $v(0)$ and of size n, where $v(0)$ is the a.s. unique $v \in \mathcal{N}$ s.t. $0 \in C_v$:

Proposition 1.3. *There are constants z_1 and C_1 such that for every $z \geq z_1$, and for every $n \geq 0$,*

$$\mathbb{P}\big(\max_{\phi \in \mathcal{SA}_n} \mu(\Gamma(\mathcal{N}, \phi)) > nz \big) \leq e^{-C_1 n z} \ .$$

2. The renormalization trick

The main idea to obtain results for Voronoi Polyominoes, i.e our Theorem 1.2, comparable to those of Cox et al. (1993) is to combine a renormalization trick with an adaptation of the chaining techniques from Cox et al. (1993) to our setting. The renormalization trick is to consider a box in $\mathbb{R}^d$ large enough so that it contains with a "large enough" probability some configuration of points which prevents a Delaunay cell to cross it completely. This will allow us to control the degrees of vertices along a Polyomino. The "large enough"probability alluded to is some percolation probability: we need that the "bad boxes" (those who can be crossed) do not percolate. In section 2.1, we explain this renormalization by showing how to cover animals with boxes and clusters of boxes. We also state the adaptation of Cox et al. (1993) to our setting. The proof of Theorem 1.2 is carried out in section 2.2, and the proof of Proposition 1.3 in section 2.3.

2.1. **Comparison with site percolation and lattice animals.** We denote by $\mathbb{G}_d$ the d-dimensional lattice with vertex set $\mathbb{Z}^d$ and edge set composed by pairs $(\mathbf{z}, \mathbf{z}')$ such that

$$|\mathbf{z} - \mathbf{z}'|_\infty = \max_{j=1,\ldots,d} |\mathbf{z}(j) - \mathbf{z}'(j)| = 1\,.$$

For $\mathbf{z}$ in $\mathbb{G}_d$ and $r > 0$, we define

$$B^r_{\mathbf{z}} = r\mathbf{z} + \prod_{k=1}^{d} [-r/2, r/2[^d\,.$$

(This is what we call a box in $\mathbb{R}^d$.) Now, we define the notions of nice and good boxes. These are notions depending on a set $\mathcal{N}$ of points in $\mathbb{R}^d$, which basically says that there are enough points of $\mathcal{N}$ inside the box so that a Voronoi tile of a point of $\mathcal{N}$ outside the box cannot "cross" the box. This will be given a precise sense thanks to Lemma 2.2.

Definition 2.1. *Define:*

$$\alpha_d = 2(4\lceil\sqrt{d}\rceil + 1)\,.$$

Let $\mathcal{N}$ be a set of points in $\mathbb{R}^d$. We say that a box $B \subseteq \mathbb{R}^d$ is $\mathcal{N}$-nice if, cutting it regularly into α_d^d sub-boxes:, each one of these boxes contains at least one point of the set $\mathcal{N}$. We say that a box B is $\mathcal{N}$-good if, cutting it regularly into $(3\alpha_d)^d$ sub-boxes, each one of these boxes contains at least one point of the set $\mathcal{N}$. When $\mathcal{N}$ is understood, we shall simply say that B is nice *(resp.* good*) if it is $\mathcal{N}$-nice (resp. $\mathcal{N}$-good). We say a box is $\mathcal{N}$-*ugly *(resp. $\mathcal{N}$-*bad*) if it is not $\mathcal{N}$-nice (resp. not $\mathcal{N}$-good).*

To count the number of points of $\mathcal{N}$ which are in a subset A of $\mathbb{R}^d$, we define:

$$|A|_{\mathcal{N}} = |A \cap \mathcal{N}|\,.$$

An animal $\mathbf{A}$ in $\mathbb{G}_d$ is a finite and connected subset of $\mathbb{G}_d$. To each real number r, and each bounded and connected subset A of $\mathbb{R}^d$, we associate 3^d different animals in $\mathbb{G}_d$:

$$\forall i = 1, \ldots, 3^d,\ \mathbf{A}^L_i(A) = \{\mathbf{z} \in \mathbb{G}_d \text{ s.t. } A \cap (r\vec{f_i}/3 + B^r_{\mathbf{z}}) \neq \emptyset\}\,,$$

where $\vec{f_1} = 0$ and $\vec{f_2}, \ldots, \vec{f}_{3^d}$ are the neighbors of $\mathbf{0}$ in $\mathbb{G}_d$. If $\mathbf{V}$ is a set of vertices of $\mathbb{G}_d$, define the border $\partial\mathbf{V}$ of $\mathbf{V}$ as the set of vertices which are not in $\mathbf{V}$ but have a neighbor in $\mathbf{V}$. We also define the following subsets of $\mathbb{R}^d$:

$$\forall i = 1, \ldots, 3^d,\ Bl^{r,i}(\mathbf{V}) := \bigcup_{\mathbf{z}\in\mathbf{V}} (r\mathbf{z} + B^r_0 + r\vec{f_i}/3)\,.$$

$$\forall i = 1, \ldots, 3^d,\ Ad^{r,i}(\mathbf{V}) := \left\{ x \in \mathbb{R}^d \text{ s.t. } \inf_{y \in Bl^{r,i}(\mathbf{V})} \|x - y\|_\infty < \frac{r}{2} \right\}\,.$$

To each fixed positive real number r, we associate 3^d site percolation schemes on $\mathbb{G}_d$ defined by:

$$\forall i = 1, \ldots, 3^d,\ X^{r,i}_{\mathbf{z}} = \mathbb{1}_{r\vec{f_i}/3 + B^r_{\mathbf{z}} \text{ is } \mathcal{N}-bad},\ \forall \mathbf{z} \in \mathbb{G}_d\,.$$

$X^{r,i}=(X^{r,i}_{\mathbf{z}})_{\mathbf{z}\in\mathbb{G}_d}$ is a collection of independent Bernoulli random variables. Of course, the comparison of μ to Lebesgue's measure (1) implies that $\mu(B^{r,i}_{\mathbf{z}})$ goes to infinity when r goes to infinity. Thus,

$$\lim_{r\to\infty}\sup_i\sup_{\mathbf{z}}\mathbb{P}(X^{r,i}_{\mathbf{z}}=1)=0\,.$$

When r is fixed, a vertex $\mathbf{z}$ is said to be (r,i)-*bad* (or simply *bad* when r and i are implicit) when $X^{r,i}_{\mathbf{z}}=1$. An (r,i)-*bad cluster* is then a maximal connected subset $\mathbf{C}$ of bad vertices of $\mathbb{G}_d$. Similarly, we define ugly vertices and ugly clusters. For any set of vertices $\mathbf{V}$, we define by $\mathbf{Cl}^{r,i}(\mathbf{V})$ the collection all (r,i)-bad clusters intersecting $\mathbf{V}$. We shall make a slight abuse of notation by writing $\mathbf{Cl}^{r,i}(\mathbf{z})$ to be the bad cluster containing $\mathbf{z}$, if there is any (otherwise, we let $\mathbf{Cl}^{r,i}(\mathbf{z})=\emptyset$).

The following lemma says essentially that the Delaunay cells cannot cross the boundaries of an ugly (or bad) cluster.

Lemma 2.2. *Fix $r>0$ and $i\in\{1,\dots,3^d\}$, assume that $\mathbf{C}$ is a (r,i)-ugly cluster in $\mathbb{G}_d$. Let Δ be any Delaunay cell of $\mathcal{D}(\mathcal{N})$. Then,*

$$\Delta\cap Bl^{r,i}(\mathbf{C})\neq\emptyset\Rightarrow\Delta\subset Ad^{r,i}(\mathbf{C})\,.$$

The same holds for bad clusters.

Proof: The idea of the proof is essentially the same as in Lemma 2.1 from Pimentel (2005). Suppose that $\Delta\cap Bl^{r,i}(\mathbf{C})\neq\emptyset$ but $\Delta\not\subset Ad^{r,i}(\mathbf{C})$. Let us define:

$$\partial^\infty\mathbf{C}=Ad^{r,i}(\mathbf{C})\setminus Bl^{r,i}(\mathbf{C})\,.$$

Notice that the boundary of an ugly cluster is composed of nice vertices, and therefore $\partial^\infty\mathbf{C}$ is composed of boxes B such that cutting B regularly into $\left(\frac{\alpha_d}{2}\right)^d$ sub-boxes, each sub-box contains at least one point of $\mathcal{N}$. This implies that one may find x_1, x_2 and y in Δ, and a box B of side length $\frac{r}{2}$ such that:

$$\begin{array}{l} y\in[x_1,x_2],\ y\in\text{central}(B),\ x_1\notin B,\ x_2\notin B\ ,\\ \text{and cutting } B \text{ regularly into } \left(\frac{\alpha_d}{2}\right)^d \text{ sub-boxes,}\\ \text{each sub-box contains at least one point of } \mathcal{N}\ , \end{array}\tag{2}$$

where central(B) is the only sub-box of B containing the center of B when one cuts B regularly into α_d^d sub-boxes. Since Δ is a Delaunay cell, there is a ball B' containing Δ and whose interior does not contain any point of $\mathcal{N}$. This ball necessarily has diameter greater than $\|x_1-x_2\|_2$, which is larger than $r/2$. Thus, there is a ball B' of diameter larger than $r/2$, which contains y and whose interior does not contain any point of $\mathcal{N}$. But one may see that any ball of diameter strictly larger than $2r\sqrt{d}/\alpha_d$ necessarily contains at least one of the sub-boxes of side length r/α_d. Notice that any ball of diameter strictly smaller than $\frac{r(\alpha_d/2-1)}{2\alpha_d}$ which contains a point in central(B) is totally included in B. Since

$$2\sqrt{d}r/\alpha_d<\frac{r(\alpha_d/2-1)}{2\alpha_d}\,,$$

and since B is a nice box, we deduce that B' contains a sub-box of B which contains a point of $\mathcal{N}$, whence a contradiction. $\square$

The following lemma transfers the problem of counting the degrees in our functional to counting the points in some boxes or clusters of bad boxes which cover the animals.

Lemma 2.3. *Let G be a finite collection of bounded and connected subsets of $\mathbb{R}^d$. Then, for any positive real number $r, t > 0$.*

$$\begin{aligned}\mathbb{P}\left[\sup_{\gamma\in\mathcal{G}}\sum_{v\in\mathcal{N}\cap\gamma} f(d_{\mathcal{N}}(v)) > t\right] \quad &\leq \\ &\sum_{i=1}^{3^d} \quad \mathbb{P}\left[\sup_{\gamma\in\mathcal{G}}\sum_{\mathbf{C}\in\mathbf{Cl}^{3r,i}(\mathbf{A}_i^{3r}(\gamma))} |Ad^{3r,i}(\mathbf{C})|_{\mathcal{N}} f(|Ad^{3r,i}(\mathbf{C})|_{\mathcal{N}}) > \frac{t}{3^d.2}\right] \\ &+\sum_{i=1}^{3^d} \quad \mathbb{P}\left[\sup_{\gamma\in\mathcal{G}}\sum_{\mathbf{z}\in\mathbf{A}_i^{3r}(\gamma)} |Bl^{3r,i}(\mathbf{z})|_{\mathcal{N}} f(|Bl^{3r,i}(\mathbf{z})|_{\mathcal{N}}) > \frac{t}{3^d.2}\right] .\end{aligned}$$

Proof: Let γ be any member of G. Recall we say a box is ugly if it is not nice, and say it is bad when it is not good. We write $\mathbf{z} \sim \mathbf{z}'$ if $\mathbf{z}$ and $\mathbf{z}'$ are two points adjacent on $\mathbb{G}_d$, and $\mathbf{z} \simeq \mathbf{z}'$ if $\mathbf{z} \sim \mathbf{z}'$ or $\mathbf{z} = \mathbf{z}'$.

$$\begin{aligned}\sum_{v\in\mathcal{N}\cap\gamma} f(d_{\mathcal{N}}(v)) \quad &= \quad \sum_{\mathbf{z}\in\mathbf{A}_1^r(\gamma)}\sum_{v\in\gamma\cap B_{\mathbf{z}}^r} f(d_{\mathcal{N}}(v)) \ , \\ &= \sum_{\substack{\mathbf{z}\in\mathbf{A}_1^r(\gamma)\\ \forall\mathbf{z}'\simeq\mathbf{z},\ B_{\mathbf{z}'}^r \text{ is nice}}} \sum_{v\in\gamma\cap B_{\mathbf{z}}^r} f(d_{\mathcal{N}}(v)) \qquad (3) \\ &\quad + \sum_{\substack{\mathbf{z}\in\mathbf{A}_1^r(\gamma)\\ \exists\,\mathbf{z}'\simeq\mathbf{z} \text{ s.t. } B_{\mathbf{z}'}^r \text{ is ugly}}} \sum_{v\in\gamma\cap B_{\mathbf{z}}^r} f(d_{\mathcal{N}}(v)) \ . \qquad (4)\end{aligned}$$

Now, remark that if there exists $\mathbf{z}' \simeq \mathbf{z} \in \mathbf{A}_1^r$ such that $B_{\mathbf{z}'}^r$ is *ugly*, then $r\mathbf{z} + B_0^{3r}$ is bad, and so there exists $i \in \{1, \ldots, 3^d\}$ such that $Bl^{3r,i}(\mathbf{z}) = 3r\mathbf{z} + B_0^{3r} + 3r\vec{f_i}/3$ is bad. Therefore,

$$(4) \quad \leq \quad \sum_{i=1}^{3^d} \sum_{\substack{\mathbf{z}\in\mathbf{A}_i^{3r}(\gamma)\\ Bl^{3r,i}(\mathbf{z}) \text{ is bad}}} \sum_{v\in\gamma\cap Bl^{3r,i}(\mathbf{z})} f(d_{\mathcal{N}}(v)) \ .$$

When v belongs to some bad box $Bl^{3r,i}(\mathbf{z})$, Lemma 2.2 shows that every Delaunay cell to which v belongs is included in $Ad^{3r,i}(\mathbf{Cl}^{3r,i}(\mathbf{z}))$. Therefore, in this case, $d_{\mathcal{N}}(v)$ is bounded

from above by $|Ad^{3r,i}(\mathbf{Cl}^{3r,i}(\mathbf{z}))|$. Since f is non-decreasing,

$$\begin{aligned}
\sum_{\substack{\mathbf{z}\in\mathbf{A}_i^{3r}(\gamma)\\ Bl^{3r,i}(\mathbf{z}) \text{ is bad}}} \sum_{v\in\gamma\cap Bl^{3r,i}(\mathbf{z})} f(d_{\mathcal{N}}(v)) &\leq \sum_{\substack{\mathbf{z}\in\mathbf{A}_i^{3r}(\gamma)\\ Bl^{3r,i}(\mathbf{z}) \text{ is bad}}} \sum_{v\in\gamma\cap Bl^{3r,i}(\mathbf{z})} f(|Ad^{3r,i}(\mathbf{Cl}^{3r,i}(\mathbf{z}))|_{\mathcal{N}})\ ,\\
&\leq \sum_{\substack{\mathbf{z}\in\mathbf{A}_i^{3r}(\gamma)\\ Bl^{3r,i}(\mathbf{z}) \text{ is bad}}} |Bl^{3r,i}(\mathbf{z})|_{\mathcal{N}} f(|Ad^{3r,i}(\mathbf{Cl}^{3r,i}(\mathbf{z}))|_{\mathcal{N}})\ ,\\
&= \sum_{\mathbf{C}\in\mathbf{Cl}^{3r,i}(\mathbf{A}_i^{3r}(\gamma))} f(|Ad^{3r,i}(\mathbf{C})|_{\mathcal{N}}) \sum_{\substack{\mathbf{z}\in\mathbf{A}_i^{3r}(\gamma)\\ \mathbf{Cl}^{3r,i}(\mathbf{z})=\mathbf{C}}} |Bl^{3r,i}(\mathbf{z})|_{\mathcal{N}}\ ,\\
&\leq \sum_{\mathbf{C}\in\mathbf{Cl}^{3r,i}(\mathbf{A}_i^{3r}(\gamma))} |Ad^{3r,i}(\mathbf{C})|_{\mathcal{N}} f(|Ad^{3r,i}(\mathbf{C})|_{\mathcal{N}})\ .
\end{aligned}$$

This last step allows us to transform a sum of dependent variables (the first one) into a sum of negatively correlated variables (the last one). This will be made clear with Lemma 4.2. We have proved that:

$$\sum_{\substack{\mathbf{z}\in\mathbf{A}_1^{r}(\gamma)\\ \exists\, \mathbf{z}'\simeq\mathbf{z} \text{ s.t. } B^r_{\mathbf{z}'} \text{ is } ugly}} \sum_{v\in\gamma\cap B^r_{\mathbf{z}}} f(d_{\mathcal{N}}(v)) \leq \sum_{i=1}^{3^d} \sum_{\mathbf{C}\in\mathbf{Cl}(\mathbf{A}_i^{3r}(\gamma))} |Ad^{3r,i}(\mathbf{C})|_{\mathcal{N}} f(|Ad^{3r,i}(\mathbf{C})|_{\mathcal{N}})\ . \tag{5}$$

Now, let us bound the term (3). We deduce from Lemma 2.2 that:

$$\begin{aligned}
\sum_{\substack{\mathbf{z}\in\mathbf{A}_1^{r}(\gamma)\\ \forall \mathbf{z}'\simeq\mathbf{z},\ B^r_{\mathbf{z}'} \text{ is } nice}} \sum_{v\in\gamma\cap B^r_{\mathbf{z}}} f(d_{\mathcal{N}}(v)) &\leq \sum_{\mathbf{z}\in\mathbf{A}_1^{r}(\gamma)} \sum_{v\in\gamma\cap B^r_{\mathbf{z}}} f(|\bigcup_{\mathbf{z}'\simeq\mathbf{z}} B^r_{\mathbf{z}'}|_{\mathcal{N}})\ ,\\
&\leq \sum_{\mathbf{z}\in\mathbf{A}_1^{r}(\gamma)} |B^r_{\mathbf{z}}|_{\mathcal{N}} f(|\bigcup_{\mathbf{z}'\simeq\mathbf{z}} B^r_{\mathbf{z}'}|_{\mathcal{N}})\ ,
\end{aligned}$$

then reasoning in the same way as to bound (4), we get:

$$\sum_{\substack{\mathbf{z}\in\mathbf{A}_1^{r}(\gamma)\\ \forall \mathbf{z}'\simeq\mathbf{z},\ B^r_{\mathbf{z}'} \text{ is nice}}} \sum_{v\in\gamma\cap B^r_{\mathbf{z}}} f(d_{\mathcal{N}}(v)) \leq \sum_{i=1}^{3^d} \sum_{\mathbf{z}\in\mathbf{A}_i^{3r}(\gamma)} |Bl^{3r,i}(\mathbf{z})|_{\mathcal{N}} f(|Bl^{3r,i}(\mathbf{z})|_{\mathcal{N}})\ . \tag{6}$$

Now, the lemma follows from inequalities (5) and (6). □

The following lemma is an adaptation of the technique of Cox et al. (1993) to control the tail of the greedy animals on $\mathbb{Z}^d$. Since the proof is quite technical, we defer it until section 4.1. It allows us to get exponential bounds for the probabilities appearing in the right hand side of the inequality in Lemma 2.3. For each positive integer $m \geq 1$ let:

$$\Phi_m = \{\mathbf{A} \text{ animal in } \mathbb{G}_d \text{ s.t. } |\mathbf{A}| \leq m,\ \mathbf{0}\in\mathbf{A}\}\ .$$

Lemma 2.4. *Let g be a nondecreasing function from $\mathbb{R}^+$ to $[1,\infty)$, and define:*

$$h(y) := yg^{-1}(y)\,,\ q(x) := h^{-1}(x)\,.$$

There exists $r_0 > 0$ such that for all $r > r_0$ there exist positive and finite constants c_5, c_6 and c_7 such that, if:

$$\liminf_{y\to\infty} \frac{l(y)}{\log y} \geq c_5,$$

then for every $n \geq m$,

$$\mathbb{P}(\sup_{\mathbf{A}\in\Phi_m} \sum_{\mathbf{C}\in\mathbf{Cl}^{r,i}(\mathbf{A})} g(|Ad^{r,i}(\mathbf{C})|_{\mathcal{N}}) > c_6 n) \leq e^{-c_7 l(q(n))}\ , \tag{7}$$

and

$$\mathbb{P}(\sup_{\mathbf{A}\in\Phi_m} \sum_{\mathbf{z}\in\mathbf{A}} g(|Bl^{r,i}(\mathbf{z})|_{\mathcal{N}}) > c_6 n) \leq e^{-c_7 l(q(n))}\ . \tag{8}$$

We shall also require the following result from Pimentel (2005):

Lemma 2.5. *There exist two finite and positive constants z_2, and c_{13} such that for every $r \geq 1$, and every $i = 1, \ldots, 3^d$,*

$$\forall z \geq z_2,\ \mathbb{P}(\max_{\mathcal{P}\in\Pi_n} |\mathbf{A}_i^r(\mathcal{P})| > zn) \leq 2e^{-c_{13}zn}\ .$$

2.2. **Proof of Theorem 1.2.** First,

$$\mathbb{E}(F_n) = \mathbb{E}[\max_{\mathcal{P}\in\Pi_n} \sum_{v\in\mathcal{P}} f(d_{\mathcal{N}}(v))] \;=\; n\int_0^\infty \mathbb{P}(\max_{\mathcal{P}\in\Pi_n} \sum_{v\in\mathcal{P}} f(d_{\mathcal{N}}(v)) > nz)\, dz\ ,$$

Let K be a positive real number, and $r \geq 1$,

$$\begin{aligned}
\mathbb{P}(\max_{\mathcal{P}\in\Pi_n} \sum_{v\in\mathcal{P}} f(d_{\mathcal{N}}(v)) > nz) \;&\leq\; \mathbb{P}(\max_{\mathcal{P}\in\Pi_n} |\mathbf{A}_1^r(\mathcal{P})| > Knz) \\
&+\; \mathbb{P}(\max_{\mathcal{P}\in\Pi_n} \sum_{v\in\mathcal{P}} f(d_{\mathcal{N}}(v)) > nz \text{ and } \max_{\mathcal{P}\in\Pi_n} |\mathbf{A}_1^r(\mathcal{P})| \leq \lfloor Knz \rfloor) \\
&\leq\; \mathbb{P}(\max_{\mathcal{P}\in\Pi_n} |\mathbf{A}^L(\mathcal{P})| > Knz) \\
&+\; \mathbb{P}(\sup_{\mathbf{A}\in\Phi_{\lfloor Knz\rfloor}} \sum_{v\in Bl^{r,1}(\mathbf{A})\cap\mathcal{N}} f(d_{\mathcal{N}}(v)) > nz)\ .
\end{aligned} \tag{9}$$

Thus,

$$\begin{aligned}\mathbb{E}[\sum_{v\in\mathcal{P}} f(d_{\mathcal{N}}(v))] &\leq \frac{1}{K}\mathbb{E}(\max_{\mathcal{P}\in\Pi_n}|\mathbf{A}^r(\mathcal{P})|) \\ &+ n\int_0^\infty \mathbb{P}(\sup_{\mathbf{A}\in\Phi_{\lfloor Knz\rfloor}} \sum_{v\in Bl^{r,1}(\mathbf{A})\cap\mathcal{N}} f(d_{\mathcal{N}}(v)) > nz)\, dz\ , \\ &\leq \frac{1}{K}(\frac{2}{c_{13}}+nz_2) \\ &+ n\int_0^\infty \mathbb{P}(\sup_{\mathbf{A}\in\Phi_{\lfloor Knz\rfloor}} \sum_{v\in Bl^{r,1}(\mathbf{A})\cap\mathcal{N}} f(d_{\mathcal{N}}(v)) > nz)\, dz\ , \end{aligned} \tag{10}$$

thanks to Lemma 2.5.

Now, let t be any positive number and m a positive integer. Using Lemma 2.3,

$$\begin{aligned} &\mathbb{P}(\sup_{\mathbf{A}\in\Phi_m} \sum_{v\in Bl^{r,1}(\mathbf{A})\cap\mathcal{N}} f(d_{\mathcal{N}}(v)) > t) \\ \leq\ & \sum_{i=1}^{3^d}\mathbb{P}\left[\sup_{\mathbf{A}\in\Phi_m}\sum_{\mathbf{C}\in\mathbf{Cl}^{3r,i}(\mathbf{A}_i^{3r}(Bl^{r,1}(\mathbf{A})))} |Ad^{3r,i}(\mathbf{C})|_{\mathcal{N}} f(|Ad^{3r,i}(\mathbf{C})|) > \frac{t}{3^d.2}\right] \\ &+\sum_{i=1}^{3^d}\mathbb{P}\left[\sup_{\mathbf{A}\in\Phi_m}\sum_{\mathbf{z}\in\mathbf{A}_i^{3r}(Bl^{r,1}(\mathbf{A}))} |Bl^{3r,i}(\mathbf{z})|_{\mathcal{N}} f(|Bl^{3r,i}(\mathbf{z})|_{\mathcal{N}}) > \frac{t}{3^d.2}\right]\ , \\ \leq\ & \sum_{i=1}^{3^d}\mathbb{P}\left[\sup_{\mathbf{A}\in\Phi_m}\sum_{\mathbf{C}\in\mathbf{Cl}^{3r,i}(\mathbf{A})} |Ad^{3r,i}(\mathbf{C})|_{\mathcal{N}} f(|Ad^{3r,i}(\mathbf{C})|) > \frac{t}{3^d.2}\right] \\ &+\sum_{i=1}^{3^d}\mathbb{P}\left[\sup_{\mathbf{A}\in\Phi_m}\sum_{\mathbf{z}\in\mathbf{A}} |Bl^{3r,i}(\mathbf{z})|_{\mathcal{N}} f(|Bl^{3r,i}(\mathbf{z})|_{\mathcal{N}}) > \frac{t}{3^d.2}\right]\ , \end{aligned} \tag{11}$$

Now, if

$$\liminf_{y\to\infty}\frac{l(y)}{\log y} \geq c_5\ ,$$

then inequality (11) and Lemma 2.4 imply:

$$\mathbb{P}(\sup_{\mathbf{A}\in\Phi_m} \sum_{v\in Bl^{r,1}(\mathbf{A})\cap\mathcal{N}} f(d_{\mathcal{N}}(v)) > c_6 m) \leq 3^d.2e^{-c_7 l(q(m))}\ .$$

Therefore, fix $K=\frac{1}{c_6}$ and $c_1=\sup\{\frac{2}{c_7},c_5\}$, and suppose that:

$$\liminf_{y\to\infty}\frac{l(y)}{\log y} \geq c_1\ .$$

This implies that $e^{-c_7 l(q(x))}$ is integrable on $\mathbb{R}^+$. Thus,

$$\mathbb{P}\Big(\sup_{\mathbf{A}\in\Phi_{\lfloor Knz\rfloor}} \sum_{v\in Bl^{r,1}(\mathbf{A})\cap\mathcal{N}} f(d_{\mathcal{N}}(v)) > nz\Big) \;\leq\; 3^d.2e^{-c_7 l(q(Knz))} \;,$$

and then,

$$\int_0^\infty \mathbb{P}\Big(\sup_{\mathbf{A}\in\Phi_{\lfloor c_6 nz\rfloor}} \sum_{v\in\bar{r}(\mathbf{A}+B_{\mathbf{0}}^1)\cap\mathcal{N}} f(d_{\mathcal{N}}(v)) > nz\Big)\, dz = O\left(\frac{1}{n}\right) \;.$$

Plugging these bounds into (9), (10) leads to the desired result. □

2.3. **Sketch of the proof of Proposition 1.3.** The proof of Proposition 1.3 is almost entirely similar to that of Theorem 1.2, so we shall only stress the differences. First, we need to see that the notion of a nice box is still a good one to control $\Gamma(\mathcal{N},\phi)$. To do that, we need to express $\Gamma(\mathcal{N},\phi)$ differently. Let $\mathcal{E}(\phi)$ be the set of edges of ϕ. Then,

$$\Gamma(\mathcal{N},\phi) = \bigcup_{e\in\mathcal{E}(\phi)} \Gamma(\mathcal{N},e) \;,$$

where:

$$\Gamma(\mathcal{N},e) = \{x\in\mathbb{R}^d \text{ s.t. } e\not\subset\mathcal{D}(\mathcal{N}\cup\{x\}\} \;.$$

To any Delaunay cell Δ of $\mathcal{D}(\mathcal{N})$ corresponds a unique ball $B(\Delta)$ in $\mathbb{R}^d$ containing Δ and whose interior does not contain any point of $\mathcal{N}$. It may be seen that:

$$\Gamma(\mathcal{N},e) = \bigcap_{\Delta\ni e} B(\Delta) \;,$$

where the intersection is taken over all Delaunay cells Δ which contain e. Notice that the ball $B(\Delta)$ is exactly the ball B appearing in the proof of Lemma 2.2. Thus, the same proof shows the following lemma.

Lemma 2.6. *Fix $r>0$ and $i\in\{0,\dots,3^d\}$, assume that $\mathbf{C}$ is an (r,i)-ugly cluster in $\mathbb{G}_d$. Let ϕ be a self-avoiding path in $\mathcal{D}(\mathcal{N})$. Then, for any vertex v in ϕ, and any edge $e\in\mathcal{E}(\phi)$ such that $v\in e$,*

$$v\in Bl^{r,i}(\mathbf{C}) \Rightarrow \Gamma(\mathcal{N},e)\subset Ad^{r,i}(\mathbf{C}) \;.$$

The same holds for bad clusters.

Lemma 2.6 shows that when $v\in Bl^{3r,i}(\mathbf{C})$, then, $\mu(\Gamma(\mathcal{N},e))\leq\mu(Ad^{3r,i}(\mathbf{C}))\leq r^d 9^d|\mathbf{C}|$. So we obtain the following analogue of Lemma 2.3.

Lemma 2.7.

$$\begin{aligned}\mathbb{P}[\sup_{\phi\in\mathcal{SA}_n}\mu(\Gamma(\mathcal{N},\phi))>t] \;&\leq\; \sum_{i=1}^{3^d}\mathbb{P}\left[\sup_{\phi\in\mathcal{SA}_n}\sum_{\mathbf{C}\in\mathbf{Cl}^{3r,i}(\mathbf{A}_i^{3r}(\phi))}|\mathbf{C}|>\frac{t}{3^d.2.(9r)^d}\right]\\ &+\; \sum_{i=1}^{3^d}\mathbb{P}\left[\sup_{\phi\in\mathcal{SA}_n}|\mathbf{A}_i^{3r}(\phi)|>\frac{t}{3^d.2.(9r)^d}\right] \;.\end{aligned}$$

Now, when r is chosen large enough (see section 4.1), Proposition 1.3 follows from Lemmas 2.5, 4.2 and inequality (22).

3. Applications

3.1. **The connectivity constant on the Delaunay triangulation.** Problems related to self-avoiding paths are connected with various branches of applied mathematics such as long chain polymers, percolation and ferromagnetism. One fundamental problem is the asymptotic behavior of the connective function κ_r, defined by the logarithm of the number $N_r(v)$ of self-avoiding paths (on a fixed graph $\mathcal{G}$) starting at vertex v and with r steps. For planar and periodic graphs subadditivity arguments yields that $r^{-1}\kappa_r(v)$ converges, when $r \to \infty$, to some value $\kappa \in (0, \infty)$ (the connectivity constant) independent of the initial vertex v. In disordered planar graphs subadditivity is lost but, if the underline graph possess some statistical symmetries (ergodicity), we may believe that the rescaled connective function still converges to some constant. When x is in $\mathbb{R}^d$, let us denote by $v(x)$ the (a.s unique) point of $\mathcal{N}$ such that $x \in C_v$. Let N_r denote the number of self-avoiding paths of length r starting at $v(0)$ on the Delaunay triangulation of a Poisson random set $\mathcal{N}$. We recall that the intensity of the Poisson random set is bounded from above and below by a constant times the Lebesgue measure on $\mathbb{R}^d$. From Theorem 1.2 we obtain a linear upper bound for the connective function of the Delaunay triangulation:

Proposition 3.1. *Let $\kappa(r) = \log N_r$. There exist positive constants z_3 and c_2 such that, for every $u \geq rz_3$,*

$$\mathbb{P}(\kappa(r) \geq u) \leq e^{-c_2 u^{1/3}} .$$

In particular,

$$\mathbb{E}(\kappa(r)) = O(r) .$$

Proof : We shall use the following intuitive inequality which is a consequence of Lemma 3.2 below.

$$N_r \leq \sup_{\mathcal{P}\in\Pi_{r-1}} \prod_{v\in\mathcal{N}\cap\mathcal{P}} d_{\mathcal{N}}(v) .$$

We now deduce the following:

$$\begin{aligned}
\mathbb{P}(N_r \geq t) &\leq \mathbb{P}(\sup_{\mathcal{P}\in\Pi_{r-1}} \prod_{v\in\mathcal{N}\cap\mathcal{P}} d_{\mathcal{N}}(v) \geq t) , \\
&= \mathbb{P}(\sup_{\mathcal{P}\in\Pi_{r-1}} \sum_{v\in\mathcal{N}\cap\mathcal{P}} \log(d_{\mathcal{N}}(v)) \geq \log t) , \\
&\leq \mathbb{P}(\sup_{\mathcal{P}\in\Pi_{r-1}} \sum_{v\in\mathcal{N}\cap\mathcal{P}} d_{\mathcal{N}}(v) \geq \log t) .
\end{aligned}$$

Now, notice that $f(x) = x$ is $c-$nice for every $c > 0$. Thus, according to Theorem 1.2, there are positive constants z_3 and c_2 such that, for all $t \geq e^{rz_3}$,

$$\mathbb{P}(N_r \geq t) \leq e^{-c_2(\log t)^{1/3}} .$$

Or, equivalently, for every $u \geq r z_3$,

$$\mathbb{P}(\kappa(r) \geq u) \leq e^{-c_2 u^{1/3}} \ .$$

This implies notably that:

$$\mathbb{E}(\kappa(r)) = O(r) \ .$$

□

Lemma 3.2. *Let $G = (V, E)$ be a graph with set of vertices V and set of edges E. Define, for any $n \in \mathbb{N}$, any $x \in V$ and $I \subset V$:*

$$\Delta_n(x, I) = \{ \ \text{s.a paths } \gamma \text{ with } n \text{ edges and s.t. } \gamma_0 = x, \ \gamma \cap I = \emptyset\} \ .$$

Define:

$$N_n(x, I) = |\Delta_n(x, I)| \ .$$

Then, for any $n \geq 1$,

$$\forall x \in V, \ \forall I \subset V, \ N_n(x, I) \leq \sup_{\gamma \in \Delta_{n-1}(x,I)} \prod_{v \in \gamma} d(v) \ ,$$

where $d(v)$ stands for the degree of the vertex v.

Proof : We shall prove the result by induction. When $n = 1$ it is obviously true. Indeed, if $x \in I$, then $N_1(x, I) = 0$ and if $x \notin I$, $N_1(x, I) \leq d(x)$. Suppose now that the result is true until $n \geq 1$. Let $x \in V$ and $I \subset V$. If $x \in I$, then $N_{n+1}(x, I) = 0$. Suppose thus that $x \notin I$. Then, denoting $u \sim x$ the fact that u and x are neighbours, and using the induction hypothesis,

$$\begin{aligned}
N_{n+1}(x, I) &= \sum_{u \sim x} N_n(u, I \cup \{x\}) \ , \\
&\leq \sum_{u \sim x} \sup_{\gamma \in \Delta_{n-1}(u, I \cup \{x\})} \prod_{v \in \gamma} d(v) \ , \\
&\leq \sum_{u \sim x} \sup_{u \sim x} \sup_{\gamma \in \Delta_{n-1}(u, I \cup \{x\})} \prod_{v \in \gamma} d(v) \ , \\
&= \sup_{u \sim x} \sup_{\gamma \in \Delta_{n-1}(u, I \cup \{x\})} d(x) \prod_{v \in \gamma} d(v) \ , \\
&= \sup_{\gamma' \in \Delta_n(x, I)} \prod_{v \in \gamma'} d(v) \ ,
\end{aligned}$$

and the induction is proved. □

3.2. **Linear passage weights of self-avoiding paths in percolation.** When $\mathcal{D}$ is a fixed Delaunay triangulation on the plane, the bond percolation model $X = \{X_e : e \in \mathcal{D}\}$ on the Delaunay triangulation, with parameter p and probability law $\mathbb{P}_p(.|\mathcal{D})$, is constructed by choosing each edge $e \in \mathcal{D}$ to be open, or equivalently $X_e = 1$, independently with probability p and closed, equivalently $X_e = 0$, otherwise. An open path is a path composed by open edges. We denote by $\mathbb{P}_p$ the probability measure obtained when $\mathcal{D}$ is the Delaunay triangulation based on $\mathcal{N}$, which is distributed like a Poisson random set on $\mathbb{R}^d$ with intensity measure μ. We denote by $\mathcal{C}_{\mathbf{0}}$ the maximal connected sugraph of $\mathcal{D}$ composed by edges e that belong to some open path starting from $v_{\mathbf{0}}$, the point of $\mathcal{N}$ whose Voronoi tile contains the origin, and we call it the open cluster. Define the critical probability:

$$\bar{p}_c(d) = \sup\left\{p \in [0,1] \,:\, \forall \alpha > 0 \sum_{n\geq 1} n^{\alpha}\mathbb{P}_p(|\mathcal{C}_{\mathbf{0}}| = n) < \infty\right\}. \tag{12}$$

It is extremely plausible that this critical probability coincides with the classical one, i.e $\sup\{p \text{ s.t. } \mathbb{P}_p(|\mathcal{C}_{\mathbf{0}}| = \infty) = 0\}$, but we are not going to prove this here. By using Proposition 3.1 to count the number of self-avoiding paths of size n, and then evaluating the probability that it is an open path, we have that there exists a positive and finite constant $B = B(d)$ such that if $p < 1/B$ then the probability of the event $\{|\mathcal{C}_{\mathbf{0}}| = n\}$ decays like $e^{-\alpha n^{1/(3d)}}$ for some constant α. Consequently:

Lemma 3.3. $0 < \frac{1}{B} \leq \bar{p}_c(d)$.

In the subsequent section we will require the following lemma on the density of open edges:

Lemma 3.4. *If $(1-p) < \bar{p}_c(d)$ then there exists constants (only depending on p) $a_1, a_2 > 0$ such that*

$$\mathbb{P}_p\Big(\exists \text{ s.a. } \gamma \text{ s.t. } |\gamma| \geq m \text{ and } \sum_{e\in\gamma} X_e \leq a_1 m\Big) \leq e^{-a_2 m}.$$

Remark 1. *We could prove Lemma 3.4 directly under $(1-p) < 1/B$ (again using Proposition 3.1) but we want to push forward optimality as much as we can.*

To prove Lemma 3.4 we shall first obtain a control on the number of boxes intersected by a self-avoiding path in $\mathcal{D}$. Recall that, in section 2.1, for fixed $L > 0$ and for each self-avoiding path γ in $\mathcal{D}$ we have defined an animal $\mathbf{A}(\gamma) := \mathbf{A}_1^L(\gamma)$ on $\mathbb{G}^d$ by taking the vertices $\mathbf{z}$ such that γ intersects $B_{\mathbf{z}}^{L/2}$. Let

$$g_r^L(\mathcal{N}) := \min\{|\mathbf{A}(\gamma)| \,:\, \text{s.a. } \gamma,\, \mathbf{v}_0 \in \gamma \text{ and } |\gamma| \geq r\}$$

and

$$G_r^L(\mathcal{N}) := \max\{|\mathbf{A}(\gamma)| \,:\, \text{s.a. } \gamma,\, \mathbf{v}_0 \in \gamma \text{ and } |\gamma| \leq r\}.$$

By Proposition 2.1 in Pimentel (2005) we have (this proposition is only stated for $d = 2$ but the method to prove it holds for all $d \geq 2$):

Lemma 3.5. *For each $L' \geq 1$ there exist finite and positive constants b_3, b_4, b_5, b_6 such that*

$$\mathbb{P}(g_r^{L'}(\mathcal{N}) < b_3 r) \leq e^{-b_4 r}\,, \tag{13}$$

and

$$\mathbb{P}(G_r^{L'}(\mathcal{N}) > b_5 r) \leq e^{-b_6 r}\,. \tag{14}$$

Proof of Lemma 3.4: Let $L > 0$, $\mathbf{z} \in \mathbb{G}^d$. In this section, we call $B_{\mathbf{z}}^{L/2}$ a *good box* if:

(i) For all $\mathbf{z}' \in \mathbb{G}^d$ with $|\mathbf{z} - \mathbf{z}'|_\infty = 2$, $B_{\mathbf{z}'}^{L/2}$ is $\mathcal{N}$-nice (recall Definition 2.1),

(ii) For all γ in $\mathcal{D}$ connecting $B_{\mathbf{z}}^{L/2}$ to $\partial B_{\mathbf{z}}^{3L/2}$ we have $\sum_{e\in\gamma} X_e \geq 1$.

Let

$$Y_{\mathbf{z}}(L) = \mathbb{1}_{B_{\mathbf{z}}^{L/2} \text{ is a good box}}\,.$$

Our first claim is: if $1 - p < \bar{p}_c(d)$ then

$$\lim_{L\to\infty} \mathbb{P}_p\big(Y_{\mathbf{z}}(L) = 1\big) = 1\,. \tag{15}$$

Since the intensity of the underlying Poisson random set is comparable with the Lebesgue measure (1), condition (i) has probability going to one as L goes to infinity. Now, denote by E_L the event that (ii) is false and (i) is true. Suppose that E_L occurs, and let γ be a path contradicting (ii). We may choose the first edge of γ as some $[v_1, v_2]$ intersecting $B_{\mathbf{z}}^{5L/2}$. Thanks to Lemma 2.2, and since (i) is true, $[v_1, v_2]$ lies in $B_{\mathbf{z}}^{L/2}$. Also, there is a point v' such that either v_1 or v_2 lies at distance at least L from v' and γ connects v_1 and v_2 to v' with closed edges. Divide $B_{\mathbf{z}}^{5L/2}$ regularly into subcubes of side length 1. This partition $\mathcal{P}_L$ has cardinality of order L^d. For each box B, let $A(B)$ be the event that there exist a vertex $v \in \mathcal{N} \cap B$, $v' \in \mathcal{N}$ such that $|v - v'| \geq L$, and a path γ in $\mathcal{D}$ from v to v' such that $\sum_{e\in\gamma} X_e = 0$. Denote by A_L the event $A(B_{\mathbf{0}}^{1/2})$. The remarks we just made imply that:

$$\mathbb{P}_p\big(E_L\big) \leq \sum_{B\in\mathcal{P}_L} \mathbb{P}_p\big(A(B)\big) \leq cL^d \mathbb{P}_p\big(A_L\big)\,.$$

Now, let us bound $\mathbb{P}_p\big(A_L\big)$. If $p = 1$, $\mathbb{P}_p\big(A_L\big) = 0$, so we suppose that $p < 1$. Let $\mathbf{B}_d$ denote the ball of center $\mathbf{0}$ and radius $3\sqrt{d}$, and let $\mathcal{E}_d$ be the set of edges of $\mathcal{D}$ lying completely inside $\mathbf{B}_d$. A simple geometric lemma (see Lemma 4.5 in section 4.2) shows that if $v \in N \cap B_0^{1/2}$, there is a path from $v_{\mathbf{0}}$ to v on $\mathcal{D}$ whose edges are in $\mathcal{E}_d$. Thus, if A_L holds and if all the edges in $\mathcal{E}_d$ are closed, then there is a closed path from $v_{\mathbf{0}}$ to some vertex v' such that $|v'| \geq L - \sqrt{2}$. Let L be larger than $2\sqrt{2}$. Using (14) (with $L' = 1$), there exist constants $a, b >$ such that, with probability greater than $(1 - e^{-aL})$, a path γ in the Poisson-Delaunay triangulation, that connects $v_{\mathbf{0}}$ to a point v' such that $|v'| \geq L - \sqrt{2}$, has at least bL edges. Now, using the FKG inequality, we have:

$$\mathbb{P}_p(A_L|\mathcal{D}) \leq \mathbb{P}_p(A_L|\{\forall e \in \mathcal{D},\ X_e = 0\},\ \mathcal{D}) \leq \frac{1}{(1-p)^{|\mathcal{E}_d|}} \left(\mathbb{P}_p(|\mathbf{C}_{\mathbf{0}}^{cl}| \geq bL|\mathcal{D}) + e^{-aL}\right)\,,$$

where $\mathbf{C}_{\mathbf{0}}^{cl}$ is the *closed* cluster containing $\mathbf{0}$. Thus:

$$\mathbb{P}_p(A_L) \leq \mathbb{E}\left[(1-p)^{-|\mathcal{E}_d|}\left(\mathbb{P}_p(|\mathbf{C}_{\mathbf{0}}^{cl}| \geq bL|\mathcal{D}) + e^{-aL}\right)\right]\,.$$

Since $\mathbf{B}_d$ is a convex set and $\mathcal{D}$ is a triangulation, a simple application of Euler's formula gives $|\mathcal{E}_d| \leq 3(|\mathcal{N} \cap \mathbf{B}_d| - 1)$. Now, notice that $|\mathcal{N} \cap \mathbf{B}_d|$ has Poisson distribution with parameter $\mu(\mathbf{B}_d)$. Thus, one can find some positive constants C_1, C_2 and C_3, depending only on μ and d, such that, for every $L > 0$:

$$\mathbb{P}(|\mathcal{N} \cap \mathbf{B}_d| \geq 3C_1 \log(L)) \leq C_2 L^{-2(d+1)} \,. \tag{16}$$

Now, using Cauchy-Schwartz inequality, and inequality (16),

$$\begin{aligned}
\mathbb{P}_p(A_L) &\leq e^{-aL}\mathbb{E}\left[(1-p)^{-|\mathcal{E}_d|}\right] \\
&\quad + \mathbb{E}\left((1-p)^{-|\mathcal{E}_d|}\ \mathbb{I}_{|\mathcal{E}_d| \geq C_1 \log(L)}\right) + (1-p)^{-C_1 \log L}\mathbb{P}_p(|\mathbf{C}^{cl}_{\mathbf{0}}| \geq bL) \,, \\
&\leq e^{-aL}\mathbb{E}\left[(1-p)^{-|\mathcal{E}_d|}\right] \\
&\quad + \sqrt{\mathbb{E}\left[(1-p)^{-2|\mathcal{E}_d|}\right]}.\sqrt{\mathbb{P}(|\mathcal{E}_d| \geq C_1 \log(L))} + L^{C_1 \log \frac{1}{1-p}}\mathbb{P}_p(|\mathbf{C}^{cl}_{\mathbf{0}}| \geq bL) \,, \\
&\leq e^{-aL}C_3 + C_3\sqrt{C_2}L^{-(d+1)} + L^{C_1 \log \frac{1}{1-p}}\mathbb{P}_p(|\mathbf{C}^{cl}_{\mathbf{0}}| \geq bL) \,,
\end{aligned}$$

where C_3 is some finite bound (depending on $p < 1$) on $\sqrt{\mathbb{E}\left[(1-p)^{-2|\mathcal{E}_d|}\right]}$. Now, if $0 < (1-p) < \bar{p}_c(d)$,

$$\begin{aligned}
L^d\mathbb{P}\big(A_L\big) &\leq L^d e^{-aL}C_3 + C_3\sqrt{C_2}L^{-1} + L^{d + C_1 \log \frac{1}{1-p}}\mathbb{P}_p(|\mathbf{C}^{cl}_{\mathbf{0}}| \geq bL) \,, \\
&\leq \sum_{n \geq bL}\left(\frac{n}{b}\right)^{d + C_1 \log \frac{1}{1-p}}\mathbb{P}_p(|\mathbf{C}^{cl}_{\mathbf{0}}| = n) + L^d e^{-aL}C_3 + C_3\sqrt{C_2}L^{-1} \,,
\end{aligned}$$

which goes to zero as L goes to infinity. This concludes the proof of (15).

Our second step is: the collection $\{Y_{\mathbf{z}}(L) : \mathbf{z} \in \mathbb{G}^d\}$ is a 5-dependent Bernoulli field. To see this, first notice that $Y_z(L) = Y_{\mathbf{z}}(L, \mathcal{N}, X)$. Thus, the claim will follow if we prove that $Y_{\mathbf{z}}(L, \mathcal{N}, X) = Y_{\mathbf{z}}(L, \mathcal{N} \cap B^{5L/2}_{\mathbf{z}}, X)$. To prove this, assume that $Y_{\mathbf{z}}(L, \mathcal{N}, X) = 0$. Then, either (i) does not hold, or (i) holds and (ii) does not. To see if (i) does not hold, it is clear that we only have to check $\mathcal{N} \cap B^{5L/2}_{\mathbf{z}}$. If (i) holds and (ii) not, we then use Lemma 2.2 to show that, under (i), inside $B^{3L/2}_{\mathbf{z}}$ the Delaunay triangulation based on $\mathcal{N}$ is the same as the one based on $\mathcal{N} \cap B^{5L/2}_{\mathbf{z}}$. Thus, if (ii) does not hold for $\mathcal{N}$, it will certainly not hold for $\mathcal{N} \cap B^{5L/2}_{\mathbf{z}}$. The same argument works to show that if $Y_{\mathbf{z}}(L, \mathcal{N}, X) = 1$ then $Y_{\mathbf{z}}(L, \mathcal{N} \cap B^{5L/2}_{\mathbf{z}}, X) = 1$.

Let

$$m_r(Y(L)) := \min\left\{\sum_{\mathbf{z} \in \mathbf{A}} Y_{\mathbf{z}} \text{ s.t. } \mathbf{A} \text{ is an animal and } |A| \geq r\right\}.$$

With (15) and 5-dependence in hands, we can chose $L_0 \geq 1$ large enough such that

$$\mathbb{P}_p(m_r(Y(L_0)) < c_1 r) \leq e^{-c_2 r} \,, \tag{17}$$

for c_1, c_2 only depending on L_0. The proof of (17) follows the same line as the proof of Lemma 4.4 in Pimentel (2005) (by first proving (17) for i.i.d. sums, and then using Liggett-Schonmann-Stacey theorem Liggett et al. (1997) to extend it to the 5-dependent case).

Let γ be a path in $\mathcal{D}$ with $v_0 \in \gamma$ and $|\gamma| \geq m$. If $B_{\mathbf{z}_1}^{L/2}$ and $B_{\mathbf{z}_2}^{L/2}$ are good boxes in $\mathbf{A}(\gamma)$, and $|\mathbf{z}_1 - \mathbf{z}_2| \geq 5$, then for each $j = 1, 2$ there is a piece of γ, say γ_j, connecting $\partial B_{\mathbf{z}_j}^{L/2}$ to $\partial B_{\mathbf{z}_j}^{3L/2}$ and with $\sum_{e \in \gamma_j} X_e \geq 1$. Since these good boxes are 5-distant, by Lemma 2.2, γ_1 and γ_2 can be assumed disjoint. This yields that $\sum_{e \in \gamma} X_e \geq 2$. By repeating this argument inductively, one gets that

$$\frac{\sum_{\mathbf{z} \in \mathbf{A}(\gamma)} Y_{\mathbf{z}}(L_0)}{11^d} \leq \sum_{e \in \gamma} X_e \,.$$

(We count good points in $\mathbb{G}^d$ that are at distance 5 from each other and so we have to divide by 11^d.) On the other hand, $\mathbf{A}(\gamma)$ has at least $k \geq g_m(\mathcal{N})$ boxes, and hence $j \geq m_k(Y(L_0))$ good boxes. Combining this together with (13) and (17), we finish the proof of Lemma 3.4.

□

3.3. **Variance bound for FPP on the Delaunay triangulation.** On any graph $\mathcal{G}$, one can define a First Passage Percolation model (FPP in the sequel) as follows. Assign to each edge e of $\mathcal{G}$ a random non-negative time $t(e)$ necessary for a particle to cross edge e. The First Passage time from a vertex u to a vertex v on $\mathcal{G}$ is the minimal time needed for a particle to go from u to v following a path on $\mathcal{G}$. This is of course a random time, and the understanding of the typical fluctuations of this times when u and v are far apart is of fundamental importance, notably because of its physical interpretation as a growth model. The model was first introduced by Hammersley and Welsh Hammersley and Welsh (1965) on $\mathbb{Z}^d$ with i.i.d passage times. Some variations on this model have been proposed and studied, see Howard (2004) for a review. One of these variations is FPP on the Delaunay triangulation, introduced in Vahidi-Asl and Wierman (1990), where the graph $\mathcal{G}$ is the Delaunay graph of a Lebesgue-homogeneous Poisson random set in $\mathbb{R}^d$. Often, this Delaunay triangulation is heuristically considered to behave like a perturbation of the triangular lattice. One important question is whether the fact that $\mathcal{G}$ itself is random affects the fluctuations of the first passage times. Since it is not already known exactly what is the right order of these fluctuations on the triangular lattice, one does not have an exact benchmark, but the best bound known up today is that the variance of the passage time between the origin and a vertex at distance n is of order $O(n/\log n)$. We are not able to show a similar bound for FPP on the Delaunay triangulation, but we can show the analogue of Kesten's bound of Kesten (1993): the above variance is at most of order $O(n)$. This improves upon the results in Pimentel (2005) and answers positively to open problem 12, p. 169 in Howard (2004).

We now precise our notations. Let ν be a probability measure on $\mathbb{R}^+$. Recall that Ω is the set of at most countable subsets of $\mathbb{R}^d$. Let π denote the Poisson measure on $\mathbb{R}^d$ with intensity μ, where μ is comparable to the Lebesgue measure on $\mathbb{R}^d$, in the sense of (1). Let $\mathcal{E}_d$ denote the set of pairs $\{x, y\}$ of points of $\mathbb{R}^d$. We endow the space $\Omega \times \mathbb{R}_+^{\mathcal{E}_d}$ with the product measure $\pi \otimes \nu^{\otimes \mathcal{E}_d}$ and each element of $\Omega \times \mathbb{R}_+^{\mathcal{E}_d}$ is denoted $(\mathcal{N}, \tau)$. This means that

$\mathcal{N}$ is a Poisson random set with intensity μ, and to each edge $e \in \mathcal{D}(\mathcal{N})$ is independently assigned a non-negative random variable $\tau(e)$ from the common probability measure ν.

The passage time $T(\gamma)$ of a path γ in the Delaunay triangulation is the sum of the passage times of the edges in γ:

$$T(\gamma) = \sum_{e \in \gamma} \tau(e) \ .$$

The first-passage time between two vertices v and v' is defined by

$$T(v, v') := T(v, v', \mathcal{N}, \tau)) = \inf\{T(\gamma) \, ; \, \gamma \in \Gamma(v, v', \mathcal{N})\} \ .$$

where $\Gamma(v, v', \mathcal{N})$ is the set of all finite paths connecting v to v'. Given $x, y \in \mathbb{R}$ we define $T(x, y) := T(v(x), v(y))$.

Remark that $(\mathcal{N}, \tau) \mapsto T(x, y)$ is measurable with respect to the completion of $\mathcal{F} \otimes \mathcal{B}(\mathbb{R}_+)^{\otimes \mathcal{E}_d}$, where $\mathcal{B}(\mathbb{R}_+)$ is the borelian tribe on $\mathbb{R}_+$ and $\mathcal{F}$ is the smallest algebra on Ω which lets the coordinate applications $\mathcal{N} \mapsto \mathbb{1}_{x \in \mathcal{N}}$ measurable. To see this, fix $x, y \in \mathbb{R}^d$ and for each $r > |x, y|$ define $T_r(x, y)$ to be the first-passage time restricted to the Delaunay graph $\mathcal{D}(\mathcal{N} \cap D_r(x, y))$, where $D_r(x, y)$ is the ball centred at x and of radius r. Then clearly T_r is measurable, since it is the infimum over a countable collection of measurable functions. On the other hand, $T_r(x, y)$ is non-increasing with r, and so $T = \lim_{r \to \infty} T_r(x, y)$ is also measurable.

We may now state the announced upper-bound on the typical fluctuations of the first passage time. We denote by $M_k(\nu)$ the $k - th$ moment of ν:

$$M_k(\nu) = \int |x|^k \, d\nu(x) \ .$$

Theorem 3.6. *For any natural number $n \geq 1$, let $\vec{n}$ denote the point $(n, 0, \ldots, 0) \in \mathbb{R}^d$. Assume that $\nu(\{0\}) < \bar{p}_c$ and that*

$$M_2(\nu) := \int x^2 \, d\nu(x) < \infty \ .$$

Then, as n tends to infinity,

$$\mathsf{Var}(T(0, \vec{n}, \tau, \mathcal{N})) = O(n) \ .$$

Before we prove Theorem 3.6, we need to state a Poincaré inequality for Poisson random sets (for a proof, see for instance Houdré and Privault (2003), inequality (2.12) and Lemma 2.3).

Proposition 3.7. *Let $F : \Omega \to \mathbb{R}$ be a square integrable random variable. Then,*

$$\mathsf{Var}_\pi(F) \leq \mathbb{E}_\pi(V_-(F)) \ ,$$

where:

$$V_-(F)(\mathcal{N}) = \sum_{v \in N} (f(\mathcal{N}) - f(\mathcal{N} \setminus \{v\}))_-^2 + \int (f(\mathcal{N}) - f(\mathcal{N} \cup \{x\}))_-^2 \, d\lambda(x) \ .$$

A geodesic between x and y, in the FPP model, is a path $\rho(x,y)$ connecting v_x to v_y and such that

$$T(x,y) = T\big(\rho(x,y)\big) = \sum_{e\in\rho(x,y)} \tau(e)\,.$$

When a geodesic exists, we shall denote by $\rho_n(\tau,\mathcal{N})$ a geodesic between 0 and $\vec{n}$ with minimal number of vertices. If there are more than one such geodesics, we select one according to some deterministic rule. We shall write $T_n(\tau,\mathcal{N})$ for $T(0,\vec{n},\tau,\mathcal{N})$.

We shall need to control the length of $\rho_n(\tau,\mathcal{N})$. The following result generalizes to every dimension the control obtained in Proposition 2.2 Pimentel (2005), but here the condition on $\nu(\{0\})$ is not optimal.

Lemma 3.8. *There exist constants a_0, C_1, C_2 and C_3 depending only on $\nu(\{0\})$ and d, and a random variable Z_n such that if $\nu(\{0\}) < \bar{p}_c(d)$, then for every n and every m,*

$$\mathbb{P}(|\rho_n(\tau,\mathcal{N})| \geq m) \leq C_2 e^{-C_1 m} + \mathbb{P}(Z_n > am)\ ,$$

and

$$\mathbb{E}(Z_n) \leq C_3 n\ .$$

Proof : For any $a>0$, and $m\in\mathbb{N}$,

$$\begin{aligned}\mathbb{P}(|\rho_n(\tau,\mathcal{N})| \geq m) &\leq \mathbb{P}(\exists \text{ s.a. } \gamma \text{ s.t. } |\gamma|\geq m \text{ and } \sum_{e\in\gamma}\tau(e)\leq am)\\ &+ \mathbb{P}(T_n(\tau,\mathcal{N}) > am)\ .\end{aligned}$$

Fix $\epsilon>0$ such that $\mathbb{P}(\tau_e>\epsilon) < \bar{p}_c$ and let $X_e = \mathbb{1}\{\tau_e > \epsilon\}$. Then $\epsilon X_e \leq \tau_e$, and consequently

$$\mathbb{P}(\exists \text{ s.a. } \gamma \text{ s.t. } |\gamma|\geq m \text{ and } \sum_{e\in\gamma}\tau(e)\leq am) \leq \mathbb{P}(\exists \text{ s.a. } \gamma \text{ s.t. } |\gamma|\geq m \text{ and } \sum_{e\in\gamma}X_e\leq a'm)\,,$$

where $a' = a/\epsilon$. Together with Lemma 3.4, this implies that

$$\mathbb{P}(\exists \text{ s.a. } \gamma \text{ s.t. } |\gamma|\geq m \text{ and } \sum_{e\in\gamma}\tau(e)\leq am) \leq C_2 e^{-C_4 m}\,.$$

Finally, in Pimentel (2005) and for $d=2$, it is shown that there is a particular path γ_n from 0 to $\vec{n}$, that is independent of τ and whose expected number of edges has is of order n. This path is constructed by walking through neighbour cells that intersect line segment $[0,\vec{n}]$ (connecting 0 to $\vec{n}$). Hence, the size of γ_n is at most the number of cells intersecting $[0,\vec{n}]$, which turns to be of order n. This construction is easily generalized to $\mathbb{R}^d$ and thus, denoting $Z_n = \sum_{e\in\gamma(0,\vec{n})}\tau(e)$ the passage time along this path, we have:

$$\mathbb{P}(T_n(\tau,\mathcal{N}) > am) \leq \mathbb{P}(Z_n > am)\ . \tag{18}$$

Using only the fact that the edge-times possess a finite moment of order 1, there is a constant C_3 such that:

$$\mathbb{E}(Z_n) = \mathbb{E}(\tau_e)\mathbb{E}(|\gamma_n|) \leq C_3 n\ .$$

$\square$

Proof of Theorem 3.6 : For any function f from $\mathbb{R}_+^{\mathcal{E}_d} \times \Omega$ to $\mathbb{R}$, we write:

$$\begin{array}{rcl} \mathbb{E}_\nu(f) & : & \left\{ \begin{array}{lcl} \Omega & \to & \mathbb{R} \\ N & \mapsto & \int f(\tau, \mathcal{N})\, d\nu^{\otimes \mathcal{E}_d}(\tau) \end{array} \right. \\ \mathsf{Var}_\nu(f) & = & \mathbb{E}_\nu(f^2) - \mathbb{E}_\nu(f)^2 \ , \end{array}$$

and for any function g in $L^2(\Omega, \pi)$,

$$\mathbb{E}_\pi(g) = \int g(\mathcal{N})\, d\pi(\mathcal{N}) \ ,$$

$$\mathsf{Var}_\pi(g) = \mathbb{E}_\pi(g^2) - \mathbb{E}_\pi(g)^2 \ .$$

We shall use the following decomposition of the variance:

$$\mathsf{Var}(f) = \mathbb{E}_\pi(\mathsf{Var}_\nu(f)) + \mathsf{Var}_\pi(\mathbb{E}_\nu(f)) \ .$$

To show that $\mathbb{E}_\pi(\mathsf{Var}_\nu(T_n)) = O(n)$ is now standard. Indeed, π-almost surely (see (2.17) and (2.24) in Kesten (1993)):

$$\mathsf{Var}_\nu(T_n) \leq 2M_2(\nu)\mathbb{E}_\nu(\rho_n) \ .$$

Thus,

$$\mathbb{E}_\pi(\mathsf{Var}_\nu(T_n)) \leq 2M_2(\nu)\mathbb{E}(\rho_n) = O(n) \ ,$$

according to Lemma 3.8. The harder part to bound is $\mathsf{Var}_\pi(\mathbb{E}_\nu(T_n))$. Let us denote by F_n the random variable $\mathbb{E}_\nu(T_n)$. We want to apply Proposition 3.7 to F_n. First note that F_n belongs to $L^2(\Omega, \pi)$: this follows from (18). We claim that:

$$\forall v \in \mathcal{N},\ (F_n(\mathcal{N}) - F_n(\mathcal{N} \setminus \{v\}))_-^2 \leq 2.3^{2(d-1)} M_2(\nu) d_{\mathcal{N}}(v)^2 \mathbb{E}_\nu(\ \mathbb{1}_{v \in \rho_n(\tau, \mathcal{N})}) \ , \tag{19}$$

and

$$\forall x \notin \mathcal{N},\ (F_n(\mathcal{N}) - F_n(\mathcal{N} \cup \{x\}))_-^2 \leq 4M_1(\nu)^2 \mathbb{E}_\nu(\ \mathbb{1}_{x \in \Gamma(\mathcal{N}, \rho_n(\tau, \mathcal{N}))}) \ . \tag{20}$$

Indeed, suppose first that v belongs to $\mathcal{N}$. If $v \notin \rho_n(\tau, \mathcal{N})$, then $\rho_n(\tau, \mathcal{N})$ is still included in $\mathcal{D}(\mathcal{N} \setminus \{v\})$. This implies that $T_n(\tau, \mathcal{N} \setminus \{v\}) \leq T_n(\tau, \mathcal{N})$. Suppose that on the contrary, $v \in \rho_n(\tau, \mathcal{N})$. Let $S_1(\mathcal{N}, v)$ be the set of edges in $\mathcal{D}(\mathcal{N})$ which belong to a Delaunay cell containing v, but which do not contain v. Define a set of edges $S_2(\mathcal{N}, v)$ containing all the edges of $\rho_n(\tau, \mathcal{N})$ that are still in $\mathcal{D}(\mathcal{N} \setminus \{v\})$, and all the edges of $S_1(\mathcal{N}, v)$. Then, $S_2(\mathcal{N}, v)$ is a set of edges in $\mathcal{D}(\mathcal{N} \setminus \{v\})$ which contains a path from 0 to $\vec{n}$. From these

considerations, we deduce that for v in $\mathcal{N}$:

$$\begin{aligned}
(F_n(\mathcal{N}) - F_n(\mathcal{N}\setminus\{v\}))_- &\leq \mathbb{E}_\nu[(T_n(\tau,\mathcal{N}) - T_n(\tau,\mathcal{N}\setminus\{v\}))_-]\ ,\\
&\leq \mathbb{E}_\nu\left[\left(\sum_{e\in S_2(\mathcal{N},v)}\tau(e) - \sum_{e\in\rho_n(\tau,\mathcal{N})}\tau(e)\right)\ \mathbb{1}_{v\in\rho_n(\tau,\mathcal{N})}\right]\ ,\\
&\leq \mathbb{E}_\nu[\sum_{e\in S_1(\mathcal{N},v)}\tau(e)\ \mathbb{1}_{v\in\rho_n(\tau,\mathcal{N})}]\ ,\\
&\leq \sqrt{\mathbb{E}_\nu\left[\left(\sum_{e\in S_1(\mathcal{N},v)}\tau(e)\right)^2\right]}\sqrt{\mathbb{E}_\nu(\ \mathbb{1}_{v\in\rho_n(\tau,\mathcal{N})})}\ ,\\
&\leq \sqrt{2M_2(\nu)}.3^{d-1}d_{\mathcal{N}}(v)\sqrt{\mathbb{E}_\nu(\ \mathbb{1}_{v\in\rho_n(\tau,\mathcal{N})})}\ ,
\end{aligned}$$

where we used Cauchy-Schwarz inequality. This gives claim (19). To see that claim (20) is true, suppose that x does not belong to $\mathcal{N}$. If x is not in $\Gamma(\mathcal{N},\rho_n(\tau,\mathcal{N}))$, obviously $\rho_n(\tau,\mathcal{N})$ is still included in $\mathcal{D}(\mathcal{N}\cup\{x\})$. Thus, $T_n(\tau,\mathcal{N}\cup\{x\}) \leq T_n(\tau,\mathcal{N})$. On the contrary, if x is in $\Gamma(\mathcal{N},\rho_n(\tau,\mathcal{N}))$, there are two special neighbors, $v_{in}(x)$ and $v_{out}(x)$ such that $\rho_n(\tau,\mathcal{N})$ still connects $v_{in}(x)$ to 0 and $v_{out}(x)$ to $\vec{n}$ in $\mathcal{D}(\mathcal{N}\cup\{x\})$. Let $S_3(\mathcal{N},x)$ be the set of edges containing all the edges of $\rho_n(\tau,\mathcal{N})$ that are still in $\mathcal{D}(\mathcal{N}\cup\{x\})$, plus the two edges $(v_{in}(x),x)$ and $(x,v_{out}(x))$. $S_3(\mathcal{N},x)$ contains a path in $\mathcal{D}(\mathcal{N}\cup\{x\})$ from 0 to $\vec{n}$. Notice that $v_{in}(x)$ and $v_{out}(x)$ depend on $\mathcal{N}$ and $(\tau(e))_{e\in\mathcal{D}(\mathcal{N})}$, and that conditionnally on $\mathcal{N}$, $\tau(v_{in}(x),x)$ and $\tau(x,v_{out}(x))$ are independent from $(\tau(e))_{e\in\mathcal{D}(\mathcal{N})}$. From this, we deduce that for x not in $\mathcal{N}$,

$$\begin{aligned}
(F_n(\mathcal{N}) &- F_n(\mathcal{N}\cup\{x\}))_-\\
&\leq \mathbb{E}_\nu[(T_n(\tau,\mathcal{N}) - T_n(\tau,\mathcal{N}\cup\{x\}))_-]\ ,\\
&\leq \mathbb{E}_\nu\Big[\Big\{\sum_{e\in S_3(\mathcal{N},x)}\tau(e) - \sum_{e\in\rho_n(\tau,\mathcal{N})}\tau(e)\Big\}\ \mathbb{1}_{x\in\Gamma(\mathcal{N},\rho_n(\tau,\mathcal{N}))}\Big]\ ,\\
&\leq \mathbb{E}_\nu[\{\tau(v_{in}(x),x) + \tau(x,v_{out}(x))\}\ \mathbb{1}_{x\in\Gamma(\mathcal{N},\rho_n(\tau,\mathcal{N}))}]\ ,\\
&\leq \mathbb{E}_\nu[\mathbb{E}_\nu\{\tau(v_{in}(x),x) + \tau(x,v_{out}(x))|(\tau(e))_{e\in\mathcal{D}(\mathcal{N})}\}\ \mathbb{1}_{x\in\Gamma(\mathcal{N},\rho_n(\tau,\mathcal{N}))}]\ ,\\
&= 2M_1(\nu)\mathbb{E}_\nu(\ \mathbb{1}_{x\in\Gamma(\mathcal{N},\rho_n(\tau,\mathcal{N}))})\ ,
\end{aligned}$$

which proves claim (20) via Jensen's inequality. Now, these two claims together with Proposition 3.7 applied to F_n give:

$$\mathsf{Var}(F_n) \leq 2.3^{2(d-1)}M_2(\nu)\mathbb{E}(\sum_{v\in\rho_n(\tau,\mathcal{N})}d_{\mathcal{N}}(v)^2) + 4M_1(\nu)^2\mathbb{E}(\lambda(\Gamma(\mathcal{N},\rho_n(\tau,\mathcal{N}))))\ . \tag{21}$$

We shall conclude using Lemma 3.8, Proposition 1.3 and Theorem 1.2. Let z_1 be as in Theorem 1.2, and a as in Lemma 3.8. Notice that $f : x \mapsto x^2$ is c-nice for any $c > 0$.

$$\begin{aligned}
\mathbb{E}[\sum_{v\in\rho_n(\tau,\mathcal{N})} f(d_{\mathcal{N}}(v))] &= n\int_0^{\infty} \mathbb{P}(\sum_{v\in\rho_n(\tau,\mathcal{N})} f(d_{\mathcal{N}}(v)) > nz)\, dz\ , \\
&\leq nz_1 + \int_{z_1}^{\infty} \mathbb{P}(\sum_{v\in\rho_n(\tau,\mathcal{N})} f(d_{\mathcal{N}}(v)) > nz)\, dz\ ,
\end{aligned}$$

From Lemma 3.8 and Theorem 1.2, we have:

$$\begin{aligned}
\mathbb{P}(\sum_{v\in\rho_n} f(d_{\mathcal{N}}(v)) > nz) &\leq \mathbb{P}(|\rho_n| > nz/z_1) + \sum_{k=0}^{nz/z_1} \mathbb{P}(F_k > nz)\ , \\
&\leq e^{-C_1(nz/z_1)} + \mathbb{P}(Z_n > anz/z_1) + \frac{nz}{z_1} e^{-c_2(c_3nz)^{1/5}}\ ,
\end{aligned}$$

and thus, using that $\mathbb{E}(Z_n) = O(n)$, we get:

$$\mathbb{E}[\sum_{v\in\rho_n(\tau,\mathcal{N})} f(d_{\mathcal{N}}(v))] = O(n)\ .$$

The proof that $\mathbb{E}(\lambda(\Gamma(\mathcal{N}, \rho_n(\tau,\mathcal{N}))) = O(n)$ is completely similar, using Proposition 1.3 and Lemma 3.8. Theorem 3.6 follows now from (21).

□

3.4. **Stabbing number.** The stabbing number $\mathrm{st}_n(\mathcal{D}(\mathcal{N}))$ of $\mathcal{D}(\mathcal{N}) \cap [0,n]^d$ is defined in Addario-Berry and Sarkar (2005) as the maximum number of Delaunay cells that intersect a single line in $\mathcal{D}(\mathcal{N}) \cap [0,n]^d$. In Addario-Berry and Sarkar (2005), the following deviation result for the stabbing number of $\mathcal{D}(N) \cap [0;n]^d$ is announced in Lemma 3, and credited to Addario-Berry, Broutin and Devroye (but the precise reference seems unavailable).

Lemma 3.9 (Addario-Berry, Broutin and Devroye)**.** *Fix $d \geq 1$. Then, there are constants $\kappa = \kappa(d)$, $K = K(d)$ such that:*

$$\mathbb{E}(\mathrm{st}_n(\mathcal{D}(\mathcal{N}))) \leq \kappa n\ ,$$

and, for any $\alpha > 0$,

$$\mathbb{P}(\mathrm{st}_n(\mathcal{D}(\mathcal{N})) > (\kappa+\alpha)n) \leq e^{-\alpha n/(K\log n)}\ .$$

The importance of Lemma 3.9 is due to the fact that it is the essential tool to prove that simple random walk on $\mathcal{D}(\mathcal{N})$ is recurrent in $\mathbb{R}^2$ and transient in $\mathbb{R}^d$ for $d \geq 3$. Here, we show that our method allows to improve Lemma 3.9 as follows.

Lemma 3.10. *Fix $d \geq 1$. Then, there are constants $\kappa = \kappa(d)$, $K = K(d)$ such that:*

$$\mathbb{E}(\mathrm{st}_n(\mathcal{D}(\mathcal{N}))) \leq \kappa n\ ,$$

and, for any $\alpha > 0$,

$$\mathbb{P}(\mathrm{st}_n(\mathcal{D}(\mathcal{N})) > (\kappa + \alpha)n) \leq e^{-\alpha n} .$$

Proof : Since the proof is very close to the proof of Proposition 1.3, we only sketch the main differences. We divide the boundary of $[0,n]^d$ into $2^d 2^{d-1} n^{d-1}$ $(d-1)$-dimensional cubes of $(d-1)$-volume 1, and call this collection S_n. For each pair of cubes (s_1, s_2) in S_n, we define $\mathrm{st}(s_1, s_2)$ as the maximum number of Delaunay cells that intersect a single line-segment going from a point in s_1 to a point in s_2. Let $V(s_1, s_2)$ be the union of those line-segments. Notice that $\mathrm{st}(s_1, s_2)$ is bounded from above by the number of points which belong to a Delaunay cell intersecting $V(s_1, s_2)$. Lemma 2.2 allows us to control those points. We obtain the following analogue of Lemma 2.3.

Lemma 3.11.

$$\begin{aligned}\mathbb{P}[\mathrm{st}(s_1,s_2) > t] \;\; &\leq \;\; \sum_{i=1}^{3^d} \mathbb{P}\left[\sum_{\mathbf{C} \in \mathbf{Cl}^{3r,i}(\mathbf{A}_i^{3r}(V(s_1,s_2)))} |\mathbf{C}|_{\mathcal{N}} > \frac{t}{3^d.2}\right] \\ &\quad + \sum_{i=1}^{3^d} \mathbb{P}\left[\sup_{\phi \in \mathcal{SA}_n} |\mathbf{A}_i^{3r}(V(s_1,s_2))|_{\mathcal{N}} > \frac{t}{3^d.2}\right] .\end{aligned}$$

Note that the cardinals of $\mathbf{A}_i^{3r}(V(s_1, s_2))$, $i = 1, \ldots, 3^d$ are of order $O(n)$, and the possible choices for the pair (s_1, s_2) is of order $O(n^{2(d-1)})$. Now, when r is chosen large enough (see section 4.1), Proposition 1.3 follows from Lemmas 2.5, 4.2 and 4.1.

□

4. Appendix

4.1. **Proof of Lemma 2.4.** It suffices to prove Lemma 2.4 for $i = 1$, so we shall omit i as a subscript or superscript. Remark first that, as already noted at the beginning of section 2.1, $p_r := \sup_i \sup_z \mathbb{P}(X_{\mathbf{z}}^{r,i} = 1)$ tends to 0 as r tends to infinity. Let us choose r_0 in such a way that

$$p_r < p_c(\mathbb{G}_d) ,$$

where $p_c(\mathbb{G}_d)$ is the critical probability for site percolation on $\mathbb{G}_d$. When $r \leq r_0$, we are in the so-called "subcritical" phase for percolation of clusters of $(\mathcal{N}, 6)$-bad boxes. From now on, we fix r to satisfy $r > r_0$ and we shall omit r as a subscript or superscript, to shorten the notations. It is well known that in this "subcritical" phase, the size of the (bad)-cluster containing a given vertex decays exponentially (see Grimmett (1989), Theorem (6.75)). For instance, there is a positive constant c, which depends only on r, such that,

$$\mathbb{P}(|\mathbf{Cl}(0)| \geq x) \leq e^{-cx} . \tag{22}$$

Remark that there is probably an exponential number of animals of size less than m, and therefore, if $\sum_{\mathbf{C} \in \mathbf{Cl}(\mathbf{A})} g(|Ad(\mathbf{C})|)$ had an exponential moment, where $g(x) = xf(x)$, it would be easy to bound the first summand in the right-hand term of the inequality

above. When $f(x)$ is larger than x, $\sum_{\mathbf{C}\in\mathbf{Cl}(\mathbf{A})} g(|Ad(\mathbf{C})|)$ surely does not have exponential moments. Therefore, we have to refine the standard argument. This refinement is a chaining technique essentially due to Cox et al. (1993) and we rely heavily on that paper.

We shall prove (7), the proof of (8) being similar and easier. We begin by stating and proving two lemmas on site percolation.

Lemma 4.1. *For all $r > r_0$ there is a constant c_9 (depending only on r) such that:*

$$\mathbb{P}(|Ad(\mathbf{Cl}(\mathbf{0}))|_{\mathcal{N}} > s) \leq \mathbb{P}_{p_r}(|Ad(\mathbf{Cl}(\mathbf{0}))|_{\mathcal{N}} > s) \leq 2e^{-c_9 s} \ .$$

where by $\mathbb{P}_{p_r}$, we mean that every site is open with the same probability

$$p_r = \sup_i \sup_z \mathbb{P}(X^{r,i}_{\mathbf{z}} = 1) < p_c(\mathbb{G}_d) \, .$$

Proof: First, we shall condition on the value of $X = (X_z)$. Let u be a positive real number.

$$\mathbb{P}(|Ad(\mathbf{Cl}(\mathbf{0}))|_{\mathcal{N}} > s|X) \leq e^{-us}\mathbb{E}(e^{u|Ad(\mathbf{Cl}(\mathbf{0}))|_{\mathcal{N}}}|X)) \ .$$

Define:

$$\partial^\infty \mathbf{Cl}(\mathbf{0}) = Ad(\mathbf{Cl}(\mathbf{0})) \setminus Bl(\mathbf{Cl}(\mathbf{0})) \ .$$

Remark that, X being fixed, $|Ad(\mathbf{Cl}(\mathbf{0}))|_{\mathcal{N}} = |\mathbf{Cl}(\mathbf{0})|_{\mathcal{N}} + |\partial^\infty \mathbf{Cl}(\mathbf{0})|_{\mathcal{N}}$ and the two summands are independent. Through the comparison to Lebesgue's measure 1, the term $|\mathbf{Cl}(\mathbf{0})|_{\mathcal{N}}$ is stochastically dominated by a sum of $|\mathbf{Cl}(\mathbf{0})|$ independent random variables, each of which is a sum of independent random variables with a Poisson distribution of parameter $\beta = \beta(r)$, conditioned on the fact that one of them at least must be zero. The term $|\partial^\infty \mathbf{Cl}(\mathbf{0})|_{\mathcal{N}}$ is stochastically dominated by a sum of at most $c_d|\mathbf{Cl}(\mathbf{0})|$ random variables (c_d only depending on d), each of which is a sum of l independent random variables with Poisson distribution, conditioned on the fact that all of them are greater than one. Obviously, the sum of l independent random variables with a Poisson distribution, conditioned on the fact that one of them at least must be zero is stochastically smaller than the sum of the same number of independent random variables with Poisson distribution, conditioned on the fact that all of them are greater than one. If Z is a Poisson random variable with parameter β, one has:

$$\mathbb{E}(e^{uZ}|Z \geq 1) = \frac{e^{\beta(e^u-1)} - e^{-\beta}}{1-e^{-\beta}} \ .$$

Therefore,

$$\mathbb{E}(e^{u|Ad(\mathbf{Cl}(\mathbf{0}))|_{\mathcal{N}}}|X) \leq \left(\frac{e^{\beta(e^u-1)}}{1-e^{-\beta}}\right)^{7|\mathbf{Cl}(\mathbf{0})|} ,$$

and thus:

$$\mathbb{P}(|Ad(\mathbf{Cl}(\mathbf{0}))|_{\mathcal{N}} > s|X) \leq e^{-us}\left(\frac{e^{\beta(e^u-1)}}{1-e^{-\beta}}\right)^{7|\mathbf{Cl}(\mathbf{0})|} .$$

Let us denote $\phi(t) = t\log t - t + 1$, $K = \frac{1}{(1-e^{-\beta})^{\frac{1}{\beta}}}$ and $K' = \phi^{-1}(2\log K)$. Remark that ϕ is a bijection from $[1,+\infty)$ onto $[0,+\infty)$. Minimizing in u gives:

$$\begin{aligned}\mathbb{P}(|Ad(\mathbf{Cl}(\mathbf{0}))|_{\mathcal{N}} > s|X) &\leq \left[Ke^{-\phi\left(\frac{s}{7\beta|\mathbf{Cl}(\mathbf{0})|}\right)}\right]^{7\beta|\mathbf{Cl}(\mathbf{0})|} \mathbb{1}_{\frac{s}{K'}\geq 7\beta|\mathbf{Cl}(\mathbf{0})|} + \mathbb{1}_{\frac{s}{K'}\leq 7\beta|\mathbf{Cl}(\mathbf{0})|}\ ,\\ &\leq e^{-\frac{7}{2}\beta|\mathbf{Cl}(\mathbf{0})|\phi\left(\frac{s}{7\beta|\mathbf{Cl}(\mathbf{0})|}\right)} \mathbb{1}_{\frac{s}{K'}\geq 7\beta|\mathbf{Cl}(\mathbf{0})|} + \mathbb{1}_{\frac{s}{K'}\leq 7\beta|\mathbf{Cl}(\mathbf{0})|}\ .\end{aligned}$$

Remark that:

$$\forall x > 1,\ \phi(x) \geq \min\{x-1, \frac{(x-1)^2}{e^2-1}\}\ .$$

Therefore, if $\frac{s}{7\beta|\mathbf{Cl}(\mathbf{0})|} \geq K' > 1$, denoting: $\bar{K} = \frac{1}{2}\min\{1-\frac{1}{K'}, \frac{(K'-1)^2}{2K'(e^2-1)}\}$,

$$\frac{7}{2}\beta|\mathbf{Cl}(\mathbf{0})|\phi\left(\frac{s}{7\beta|\mathbf{Cl}(\mathbf{0})|}\right) \geq \bar{K}s\ . \tag{23}$$

This leads to:

$$\mathbb{P}(|Ad(\mathbf{Cl}(\mathbf{0}))|_{\mathcal{N}} > s|X) \leq e^{-\bar{K}s} + \mathbb{1}_{\frac{s}{K'}\leq 7\beta|\mathbf{Cl}(\mathbf{0})|}\ . \tag{24}$$

Therefore, denoting $c_9 = \min\{\bar{K}, \frac{c}{7\beta K'}\}$, we get the desired result (notice that β and K' depend on r). □

Lemma 4.2. *Let Λ be a finite subset of $\mathbb{Z}^d$. If f is an increasing function from $\mathbb{N}$ to $[1,+\infty[$,*

$$\mathbb{E}(\Pi_{\mathbf{C}\in\mathbf{Cl}(\Lambda)}f(|\mathbf{C}|)) \leq \mathbb{E}(f(|\mathbf{Cl}(\mathbf{0})|))^{|\Lambda|}\ .$$

Proof : Let us recall Reimer's inequality (cf. Grimmett Grimmett (1989) p.39). Let n be a positive integer, let $\mathbf{B}(n) = \mathbb{Z}^2 \cap [-n,n]^d$ and define $\Gamma = \{0,1\}^{\mathbf{B}(n)}$. For $\omega \in \Gamma$ and $\mathbf{K} \subset \mathbf{B}(n)$, define the cylinder event $C(\omega,\mathbf{K})$ generated by ω on $\mathbf{K}$ by:

$$C(\omega,\mathbf{K}) = \{\omega' \in \Gamma \text{ s.t. } \omega'_i = \omega_i \forall i \in \mathbf{K}\}\ .$$

If A and B are two subsets of Γ, define their disjoint intersection $A\Box B$ as follows:

$$A\Box B = \{\omega \in \Gamma \text{ s.t. } \exists \mathbf{K} \subset \mathbf{B}(n),\ C(\omega,\mathbf{K}) \subset A \text{ and } C(\omega,\mathbf{K}^c) \subset B\}\ .$$

Reimer's inequality states that:

$$\mathbb{P}(A\Box B) \leq \mathbb{P}(A)\mathbb{P}(B)\ .$$

Remark that $\Box$ is a commutative and associative operation, and that, for any l subsets $A_1,\ldots,A_l$ of Γ,

$$\begin{aligned}A_1\Box\ldots\Box A_l &= \Big\{\omega \in \Gamma \text{ s.t. } \exists \mathbf{K}_1,\ldots,\mathbf{K}_l \text{ disjoint subsets of } \mathbf{B}(n),\\ &\qquad \bigcup_{i=1}^{l}\mathbf{K}_i = \mathbf{B}(n) \text{ and } C(\omega,\mathbf{K}_i) \subset A_i \forall i = 1,\ldots,l\Big\}\ .\end{aligned}$$

Now take n large enough so that $\Lambda \subset \mathbf{B}(n)$. Let $l = |\Lambda|$, and order the elements of $\Lambda = \{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_l\}$. For any $\mathbf{x}$ in $\mathbf{B}(n)$, and any $\omega \in \Gamma$, let $\mathbf{C}_n(\mathbf{x}, \omega)$ be the closed cluster (for the configuration ω) in $\mathbf{B}(n)$ containing $\mathbf{x}$, which is empty if $\omega(\mathbf{x}) = 1$. Define:

$$\forall i,\ \mathbf{C}_i(\omega) = \begin{cases} \mathbf{C}_n(\mathbf{x}_i, \omega) & \text{if } \mathbf{x}_i \notin \bigcup_{k=1}^{i-1} \mathbf{C}_n(\mathbf{x}_k, \omega) \\ \emptyset & \text{else.} \end{cases}$$

Let $k_1, \ldots, k_l$ be nonnegative integers and:

$$A_i = \{\omega \text{ s.t. } |\mathbf{C}_i(\omega)| \geq k_i\},\ \forall 1 \leq i \leq l\ ,$$

$$\widetilde{A}_i = \{\omega \text{ s.t. } |\mathbf{C}_n(\mathbf{x}_i, \omega)| \geq k_i\},\ \forall 1 \leq i \leq l\ .$$

Then, we claim that:

$$\bigcap_{i=1}^{l} A_i \subset \widetilde{A}_1 \square \ldots \square \widetilde{A}_l\ .$$

Indeed, let $\omega \in \bigcap_{i=1}^{l} A_i$. Then, for every $i = 1 \ldots, l-1$, $C(\omega, \mathbf{C}_i(\omega)) \subset A_i$. Furthermore, $C(\omega, \mathbf{B}(n) \setminus \bigcup_{i=1}^{l-1} \mathbf{C}_i(\omega)) \subset A_l$. This shows that $\omega \in \widetilde{A}_1 \square \ldots \square \widetilde{A}_l$, and proves the claim above. Therefore, using Reimer's inequality (which is equivalent to BK inequality in this context),

$$\mathbb{P}(\bigcap_{i=1}^{l} A_i) \leq \Pi_{i=1}^{l} \mathbb{P}(\widetilde{A}_i)\ . \tag{25}$$

Now, let f be an increasing function from $\mathbb{R}^+$ to $[1, \infty)$. Define f_1 as follows:

$$f_1(0) = 1 \text{ and } \forall k \geq 1,\ f_1(k) = f(k)\ .$$

Denote by $\{\alpha_0 = 1 < \alpha_1 < \ldots < \alpha_j < \ldots\}$ the range of f_1. Define:

$$k_j = \inf\{k \text{ s.t. } f(k) \geq \alpha_j\},\ \forall j \in \mathbb{N}\ ,$$

$$A_{i,j} = \{\omega \text{ s.t. } |\mathbf{C}_i(\omega)| \geq k_j\} = \{\omega \text{ s.t. } f_1(|\mathbf{C}_i(\omega)|) \geq \alpha_j\}\ ,$$

and

$$\widetilde{A}_{i,j} = \{\omega \text{ s.t. } |\mathbf{C}_n(\mathbf{x}_i, \omega)| \geq k_j\} = \{\omega \text{ s.t. } f_1(|\mathbf{C}_n(\mathbf{x}_i, \omega)|) \geq \alpha_j\}\ .$$

By convention, set $\alpha_{-1} = 0$ and define $a_j = \alpha_j - \alpha_{j-1}$. We can write:

$$f_1(|\mathbf{C}_i(\omega)|) = \sum_{j \in \mathbb{N}} (\alpha_j - \alpha_{j-1})\ \mathbb{1}_{A_{i,j}} = \sum_{j \in \mathbb{N}} a_j\ \mathbb{1}_{A_{i,j}}\ .$$

Define $\mathbf{C}_n(\Lambda)$ as the set of nonempty components $\mathbf{C}_n(\mathbf{x}_i,\omega)$, for $i \in \{1,\ldots,l\}$. Since $f_1(0)=1$, we can write:

$$\begin{aligned}
\mathbb{E}(\Pi_{\mathbf{C}\in\mathbf{C}_n(\Lambda)} f(|\mathbf{C}|)) &= \mathbb{E}(\Pi_{\mathbf{C}\in\mathbf{C}_n(\Lambda)} f_1(|\mathbf{C}|)) \ , \\
&= \mathbb{E}(\Pi_{i=1}^l f_1(|\mathbf{C}_i(\omega)|)) \ , \\
&= \mathbb{E}(\Pi_{i=1}^l \sum_{j\in\mathbb{N}} a_j \ \mathbb{1}_{A_{i,j}}) \ , \\
&= \sum_{j_1,\ldots,j_l} a_{j_1}\ldots a_{j_l} \mathbb{E}(\Pi_{i=1}^l \ \mathbb{1}_{A_{i,j_i}}) \ , \\
&\le \sum_{j_1,\ldots,j_l} a_{j_1}\ldots a_{j_l} \Pi_{i=1}^l \mathbb{P}\left[\bigcap_{i=1}^l \widetilde{A}_{i,j_i}\right] \ , \\
&= \Pi_{i=1}^l \sum_{j\in\mathbb{N}} a_j \mathbb{P}(\widetilde{A}_{i,j_i}) \ , \\
&= \Pi_{i=1}^l \mathbb{E}(f_1(|\mathbf{C}_n(\mathbf{x}_i,\omega)|)) \ , \\
&\le \Pi_{i=1}^l \mathbb{E}(f(|\mathbf{C}_n(\mathbf{x}_i,\omega)|)) \ ,
\end{aligned}$$

where the first inequality follows from (25). Finally, we may let n tend to infinity, and conclude by Lebesgue's monotone convergence theorem. □

We may now complete the proof of Lemma 2.4. we fix m and n two integers such that $n \ge m$. First, we get rid of the variables $|Ad(\mathbf{C})|_N$ which are greater than $q(n)$. Remark that, denoting $\Lambda_m^d = [-m,m]^d \cap \mathbb{G}_d$,

$$\max_{\mathbf{A}\in\Phi_m} \sum_{\mathbf{C}\in\mathbf{Cl}(r\mathbf{A})} g(|Ad(\mathbf{C})|_{\mathcal{N}}) \le \sum_{\mathbf{z}\in\Lambda_m^d} g(|Ad(\mathbf{Cl}(\mathbf{z}))|_{\mathcal{N}}) \ .$$

Then,

$$\begin{aligned}
\mathbb{P}(\exists\, \mathbf{z}\in\Lambda_m^d \text{ s.t. } g(|Ad(\mathbf{Cl}(\mathbf{z}))|_{\mathcal{N}}) > q(n)) &\le (2m+1)^d \mathbb{P}_{p_r}(g(|Ad(\mathbf{Cl}(\mathbf{0}))|_{\mathcal{N}}) > \gamma(m)) \ , \\
&\le (2m+1)^d \mathbb{P}_{p_r}(|Ad(\mathbf{Cl}(\mathbf{0}))|_{\mathcal{N}} > l(q(n))) \ .
\end{aligned}$$

Denote by c_9 the constant of Lemma 4.1. For every m:

$$\mathbb{P}(\exists\, z\in\Lambda_m^d \text{ s.t. } g(|Ad(\mathbf{Cl}(\mathbf{z}))|_{\mathcal{N}}) > q(n)) \le (m+1)^d e^{-c_9 l(q(n))} \ . \tag{26}$$

Now, we want to bound from above the following probability:

$$\mathbb{P}(\max_{\mathbf{A}\in\Phi_m} \sum_{\mathbf{Cl}(\mathbf{0})\in\mathbf{Cl}(\mathbf{0})(r\mathbf{A})} g(|Ad(\mathbf{Cl}(\mathbf{0}))|_{\mathcal{N}}) \ \mathbb{1}_{\forall \mathbf{z}\in\Lambda_m^d,\, g(|Ad(\mathbf{Cl}(\mathbf{z}))|_{\mathcal{N}})\le q(n)}) \ .$$

First, Lemma 1 in Cox et al. (1993) (Lemma 4.3 below) remains true in our setting, without any modification (just remark that the condition $l \le n$ in their lemma is in fact not needed).

Then, we define the following box in $\mathbb{G}_d$, centered at $\mathbf{x} \in \mathbb{G}_d$:

$$\Lambda(\mathbf{x}, l) = \{(x_1 + k_1, \ldots, x_d + k_d) \in \mathbb{G}_d \text{ s.t. } (k_1, \ldots, k_d) \in [-l, l]^d\} .$$

Lemma 4.3. *Let $\mathbf{A}$ be a lattice animal of $\mathbb{G}_d$ containing $\mathbf{0}$, of size $|\mathbf{A}| = m$ and let $1 \leq l$. Then, there exists a sequence $\mathbf{x}_0 = 0, \mathbf{x}_1, \ldots, \mathbf{x}_h \in \mathbb{G}_d$ of $h+1 \leq 1+(2m-2)/l$ points such that*

$$\mathbf{A} \subset \bigcup_{i=0}^{h} \Lambda(l\mathbf{x}_i, 2l),$$

and

$$|\mathbf{x}_{i+1} - \mathbf{x}_i|_\infty \leq 1, \qquad 0 \leq i \leq h-1 .$$

Always following Cox et al. (1993), we shall use Lemma 4.3 at different "scales" k, covering a lattice animal by $1 + (2m-2)/l_k$ boxes of length $4l_k + 1$. We shall choose l_k later. For any animal $\mathbf{A}$ and $0 < L, R < \infty$ define:

$$S(L, R; \mathbf{A}) = \sum_{\mathbf{C} \in \mathbf{Cl}(r\xi)} g(|Ad(\mathbf{C})|_{\mathcal{N}}) \; \mathbb{1}_{L \leq g(|Ad(\mathbf{C})|_{\mathcal{N}}) < R} .$$

Suppose that c_0 and $(t(n,k))_{k \leq \log_2 q(n)}$ are positive real numbers such that:

$$\sum_{k \leq \log_2 q(n)} 2^k t(n,k) \leq c_0 n . \tag{27}$$

We shall choose these numbers later. Let c' be a positive real number to be fixed later also, and define $a = 1 + c'c_0$,

$$\begin{aligned} &\mathbb{P}(\exists\ \mathbf{A} \in \Phi_m \text{ with } S(0, q(n); \mathbf{A}) > an) \\ \leq &\sum_{\substack{k \geq 0 \\ 2^k \leq q(n)}} \mathbb{P}(\exists\ \mathbf{A} \in \Phi_m \text{ with } S(2^k, 2^{k+1}; \mathbf{A}) > c't(n,k)2^k) . \end{aligned}$$

Now, fix k for the time being, let $\mathbf{A}$ be an animal of size at most m, containing $\mathbf{0}$, let $\mathbf{x}_0 = \mathbf{0}, \mathbf{x}_1, \ldots, \mathbf{x}_h$ be as in Lemma 4.3 and define $\Lambda_{m,k} := \bigcup_{i \leq h} \Lambda(l_k \mathbf{x}_i, 2l_k)$. Clearly,

$$\begin{aligned} &S(2^k, 2^{k+1}; \mathbf{A}) \\ &\leq 2^{k+1} \left(\text{number of } \mathbf{C} \in \mathbf{Cl}\left(\Lambda_{n,k}\right) \text{ with } g(|Ad(\mathbf{C})|_{\mathcal{N}}) \geq 2^k\right) . \end{aligned}$$

Lemma 4.4. *There exists $c_{10} \in (0, \infty)$ such that for all $t \geq 3|\Lambda| e^{-c_9 s}$*

$$\mathbb{P}\left(\sum_{\mathbf{C} \in \mathbf{Cl}(\Lambda)} \mathbb{1}_{g(|Ad(\mathbf{C})|_{\mathcal{N}}) > 2^k} > t \right) \leq e^{-c_{10} t} .$$

Proof: Let L be a positive real number. Notice that, conditionally on X, $(|Ad(\mathbf{C})|)_{\mathbf{C}\in\mathbf{Cl}(\Lambda)}$ are independent. Thus,

$$\mathbb{P}\left(\sum_{\mathbf{C}\in\mathbf{Cl}(\Lambda)} \mathbb{1}_{|Ad(\mathbf{C})|_{\mathcal{N}}>s} > t\right) \;\leq\; e^{-Lt}\mathbb{E}\left(\Pi_{\mathbf{C}\in\mathbf{Cl}(\Lambda)}\mathbb{E}(e^{\lambda\ \mathbb{1}_{|Ad(\mathbf{C})|_{\mathcal{N}}>s}}|X)\right)\ .$$

But,

$$\begin{aligned}\mathbb{E}(e^{L\ \mathbb{1}_{|Ad(\mathbf{C})|_{\mathcal{N}}>s}}|X) &= e^{\lambda}\mathbb{P}(|Ad(\mathbf{C})|_{\mathcal{N}} > s|X) + (1-\mathbb{P}(|Ad(\mathbf{C})|_{\mathcal{N}} > s|X))\ ,\\ &= 1+\mathbb{P}(|Ad(\mathbf{C})|_{\mathcal{N}} > s|X)(e^{L}-1)\ .\end{aligned}$$

From equation (24), we know that:

$$\mathbb{P}(|Ad(\mathbf{C})|_{\mathcal{N}} > s|X) \leq e^{-\bar{K}s} + \ \mathbb{1}_{\frac{s}{K'}\leq 7\beta|\mathbf{C}|}\ .$$

Remark that the right-hand side of this inequality is an increasing function of $|\mathbf{C}|$. Using Lemma 4.2, we deduce:

$$\begin{aligned}&\mathbb{P}\left(\sum_{\mathbf{C}\in\mathbf{Cl}(\Lambda)} \mathbb{1}_{|Ad(\mathbf{C})|_{\mathcal{N}}>s} > t\right)\\ \leq\ & e^{-Lt}\left(1+(e^{L}-1)\left[e^{-\bar{K}s}+\mathbb{P}_{p_r}(|\mathbf{Cl}(\mathbf{0})| > \frac{s}{7\beta K'})\right]\right)^{|\Lambda|}\ .\end{aligned}$$

Using inequality (22), and recalling that $c_9=\inf\{\bar{K},\frac{c}{7\beta K'}\}$,

$$\begin{aligned}\mathbb{P}\left(\sum_{\mathbf{C}\in\mathbf{Cl}(\Lambda)} \mathbb{1}_{|Ad(\mathbf{C})|_{\mathcal{N}}>s} > t\right) &\leq e^{-Lt}\left(1+2(e^{\lambda}-1)e^{-c_9 s}\right)^{|\Lambda|}\ ,\\ &\leq e^{-Lt}e^{2|\Lambda|(e^{L}-1)e^{-c_9 s}}\ ,\end{aligned}$$

Now, if $t\geq 2|\Lambda|e^{-c_9 s}$, minimizing over L gives:

$$\mathbb{P}\left(\sum_{\mathbf{C}\in\mathbf{Cl}(\Lambda)} \mathbb{1}_{g(|Ad(\mathbf{C})|_{\mathcal{N}})>2^k} > t\right) \leq e^{t-t\log\frac{t}{2|\Lambda|e^{-c_9 s}}-2|\Lambda|e^{-c_9 s}} = e^{-2|\Lambda|e^{-c_9 s}\phi\frac{t}{2|\Lambda|e^{-c_9 s}}}\ .$$

Suppose now that $t\geq 3|\Lambda|e^{-c_9 s}$. Using the same argument which led to (23), we get that there exists $c_{10}>0$, depending only on r, such that:

$$\forall\ t\geq 3|\Lambda|e^{-c_9 s},\ \mathbb{P}\left(\sum_{\mathbf{C}\in\mathbf{Cl}(\Lambda)} \mathbb{1}_{g(|Ad(\mathbf{C})|_{\mathcal{N}})>2^k} > t\right) \leq e^{-c_{10}t}\ .$$

□

Remark that:

$$|\Lambda_{m,k}| \leq \frac{2m}{l_k}(4l_k+1)^2 \leq 50ml_k \ .$$

Suppose now that

$$t(n,k) \geq 150nl_k e^{-c_9 l(2^k)} \ . \tag{28}$$

The number of choices for $\mathbf{0} = \mathbf{x}_0, \mathbf{x}_1, \dots, \mathbf{x}_h$ in Lemma 4.3 is at most 9^h. Therefore,

$$\mathbb{P}(\exists\ \mathbf{A} \in \Phi_m \text{ with } S(2^k, 2^{k+1}; \mathbf{A}) > ct(n,k)2^{k+1}) \leq 9^{\frac{2m}{l_k}} e^{-c_{10}c' t(n,k)} \ .$$

In view of the last inequality and (26), we would be happy if we had

$$t(n,k) \geq \bar{c}l(q(n))$$

and

$$c_{10}c't(n,k) \geq (2\log 9).\frac{2m}{l_k} \ .$$

Of course, we still have to check conditions (27) and (28), and we can choose c' as large as we need. A natural way to proceed is first to choose l_k in such a way that the right-hand side in (28) is proportional to $\frac{m}{l_k}$, and then to take large enough. Recall that $n \geq m$. Thus, define:

$$l_k = \left\lceil e^{\frac{c_9}{2}l(2^k)} \right\rceil \ .$$

Choose

$$t(n,k) = \max\{150ml_k e^{-c_9 l(2^k)}, l(q(n))\} \ .$$

Let c' be such that:

$$c' \geq \frac{4\log 9}{150c_{10}} \ .$$

This ensures that:

$$c'c_{10}t(n,k) \geq (4\log 9)nl_k e^{-c_9 l(2^k)} \geq (2\log 9)\frac{2n}{l_k} \ .$$

Therefore,

$$\mathbb{P}(\exists\ \mathbf{A} \in \Phi_m \text{ with } S(2^k, 2^{k+1}; \mathbf{A}) > c't(n,k)2^{k+1}) \leq e^{-c_{10}c't(n,k)/2} \ . \tag{29}$$

Condition (28) is trivially verified from the definition of $t(n,k)$. Now, let us check condition (27). Now, assume that f is $\frac{4}{c_9}$-nice. Then, using the definition of l_k,

$$\sum_{k \leq \log_2 q(n)} 2^{k+1}150nl_k e^{-c_9 l(2^k)} \leq 300n \sum_k e^{\log(2^k) - \frac{c_9}{2}l(2^k)} \ .$$

By our assumption,

$$\limsup_{k\to\infty} \frac{e^{\log(2^k) - \frac{c_9}{2}l(2^k)}}{(2^{\frac{1}{3}})^k} \leq 1 \ ,$$

and therefore,

$$\Sigma := \sum_k e^{\log(2^k) - \frac{c_9}{2}l(2^k)} < \infty \ .$$

On the other hand,

$$\sum_{k \leq \log_2 q(n)} 2^{k+1} l(q(n)) \leq 4q(n) l(q(n)) \leq 4n \ ,$$

by definition of $q(n)$. Therefore, condition (27) is checked with

$$c_0 = 300\Sigma + 4 \ .$$

Remark also that $g(x) \geq x$, and hence $l(y) \leq y$. Therefore:

$$n \leq (q(n)+1) l(q(n)+1) \leq (q(n)+1)^2 \ ,$$

$$q(n) \geq m^{1/2} - 1 \ ,$$

and thus, if

$$\liminf_{y \to \infty} \frac{l(y)}{\log y} \geq \frac{(4d+1)}{c_9} \tag{30}$$

then

$$d \log(m+1) - \frac{c_9}{2} l(q(n)) \xrightarrow[n \to \infty]{} -\infty \ ,$$

and therefore, there exist a constant c_{11} such that:

$$\forall n \geq m, \ (m+1)^d e^{-c_9 l(q(n))} \leq c_{11} e^{-\frac{c_9}{2} l(q(n))} \ .$$

Since $q(n) \leq n$, we get in the same way that there is a constant c_{12} such that:

$$\sum_{k \leq \log_2 q(n)} e^{-c_{10} c t(n,k)/2} \leq \log_2 q(n) e^{-c_{10} c l(q(n))/2} \leq c_{12} e^{-c_{10} c l(q(n))/4} \ ,$$

provided that (30) holds. Therefore, we define $c_5 = \sup\{\frac{6}{c}, \frac{4d+1}{c_9}\}$, and we suppose that

$$\liminf_{y \to \infty} \frac{l(y)}{\log y} \geq c_5 \ . \tag{31}$$

Then, for $c_6 = a = 1 + c c_0$, there exists a positive constant c_7 such that:

$$\begin{aligned} \mathbb{P}(\sup_{\mathbf{A} \in \Phi_m} \sum_{\mathbf{C} \in \mathbf{Cl}(r\xi)} g(|Ad(\mathbf{Cl}(\mathbf{0}))|_{\mathcal{N}}) > c_6 n) \ &\leq \ (m+1)^2 e^{-\bar{c} l(q(n))} \\ &\quad + \ \sum_{k \leq \log_2 q(n)} e^{-c_{10} c t(n,k)/2} \ , \\ &\leq \ e^{-c_7 l(q(n))} \ , \end{aligned}$$

where the first inequality follows from (26) and (29). This concludes the proof of Lemma 2.4. $\square$

4.2. **A simple geometric lemma.**

Lemma 4.5. *Let $\mathcal{N}$ be a locally finite set of points in $\mathbb{R}^d$ and $\mathcal{D}$ the Delaunay triangulation based on $\mathcal{N}$. Let u and v be two distinct points in $\mathcal{N}$. Then, there is a path on $\mathcal{D}$ going from v to u and totally included in the (closed) ball of center u and radius $|u-v|$.*

Proof : First we show that there is a neighbour w of v in $\mathcal{D}$ inside the (closed) ball of diameter $[u,v]$. To see this, let us define by $(B_\alpha)_{\alpha\in[0,1]}$ the collection of euclidean balls such that B_α has diameter $[v,x_\alpha]$, where $x_\alpha = v+\alpha(u-v)$. Notice that $B_\alpha \subset B_{\alpha'}$ as soon as $\alpha \leq \alpha'$. Define:

$$\alpha_0 := \min\{\alpha \in [0,1] \text{ s.t. } \exists w' \in \mathcal{N} \setminus \{v\} \cap B_\alpha\} \ .$$

This is indeed a minimum because $\mathcal{N}$ is locally finite. Notice also that the set is non-empty since it contains $\alpha = 1$. Thus the interior of B_{α_0} does not intersect $\mathcal{N}$, but ∂B_{α_0} contains (at least) two points of $\mathcal{N}$ (including v). This implies that there is a point on the sphere ∂B_{α_0} which is a neighbour of v. So we have proved that there is a neighbour w_1 of v inside the ball of diameter $[u,v]$. Then, as long as the neighbour obtained is different from u, we may iterate this construction to get a sequence of neighbours $w_0 = v, w_1, w_2, \ldots$ such that w_{i+1} belongs to the ball of diameter $[u,w_i]$. All these balls are included in the ball of center u and radius $|u-v|$. Since $\mathcal{N}$ is locally finite, this construction has to stop at some k, when the condition that w_k is distinct from u is no longer satisfied. Then the desired path is constructed. □

Acknowledgements

Raphaël Rossignol would like to thank Pierre Calka for numerous helpful discussions.

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